Lemma 4.2 Let $f : D \to \mathbb{R}$ with c in D. Make the following suppositions:

1. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{Q} \cap D$ converging to c, then $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(c)$;

and

2. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in $(\mathbb{R} \setminus \mathbb{Q}) \cap D$ converging to c, then $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(c)$.

Then f is continuous at c.

Proof. For contradiction suppose f is not continuous at c.

not continuous mean

(**) $\exists\, \varepsilon > 0\ \forall\, \delta > 0\ \exists\, x \in D$ and $|x - c| < \delta$ s.t. $|f(x) - f(c)| \geq \varepsilon$

use the fact that f is not continuous to construct a convergent sequence $x_n \to c$

Chose $\varepsilon > 0$ as in (**).

for $\delta = \frac{1}{n}$ there is $x_n \in D$ with $|x_n - c| < \frac{1}{n}$ s.t. (such that) $\boxed{|f(x_n) - f(c)| \geq \varepsilon}$.

Thus there exists $\varepsilon > 0$ and $x_n \to c$ such that $|f(x_n) - f(c)| \geq \varepsilon$.

Since x_n is either ration or irrational then define

$$A = \{ n \in \mathbb{N} : x_n \in \mathbb{Q} \} \quad \text{and} \quad B = \{ n \in \mathbb{N} : x_n \in \mathbb{R} \setminus \mathbb{Q} \}$$

We know $A \cap B = \emptyset$ and $A \cup B = \mathbb{N}$.

Therefore either A or B or both are infinite.

Case A is infinite. Then there is a subsequence $x_{n_k} \in \mathbb{Q}$

Since $x_n \to c$ then $x_{n_k} \to c$ as $k \to \infty$.

Since $x_{n_k} \in D$ then by the hypothesis

1. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{Q} \cap D$ converging to c, then $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(c)$;

implies $f(x_{n_k}) \to f(c)$. But that contradicts $|f(x_{n_k}) - f(c)| \geq \varepsilon$.

Therefore, Case A being infinite could not happen.

Case B is infinite. Then there is a subsequence $x_{n_k} \in \mathbb{R}\setminus\mathbb{Q}$

Since $x_n \to c$ then $x_{n_k} \to c$ as $k \to \infty$.

Since $x_{n_k} \in D$ then by the hypothesis

2. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in $(\mathbb{R}\setminus\mathbb{Q}) \cap D$ converging to c, then $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(c)$.

implies $f(x_{n_k}) \to f(c)$. But that contradicts $|f(x_{n_k}) - f(c)| \geq \varepsilon$.

Therefore, case B being infinite could not happen.

But then neither A nor B is infinite which contradicts that $A \cup B = \mathbb{N}$.

Consequently f is continuous at c.

<u>Examples</u> Let $f:(0,\infty) \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R}\setminus\mathbb{Q} \\ \frac{1}{n} & \text{if } x \in \mathbb{Q} \text{ where } x = \frac{m}{n} \text{ in lowest terms with } m,n \in \mathbb{N}. \end{cases}$$

Claim f is continuous at every irrational number and f is discontinuous at every rational number.

Proof that f is continuous at every irrational number

by the theorem, it is enough to consider sequences in $\mathbb{Q}$ separately from sequences in $\mathbb{R}\setminus\mathbb{Q}$.

1. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{Q} \cap D$ converging to c, then $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(c)$;

and

2. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in $(\mathbb{R}\setminus\mathbb{Q}) \cap D$ converging to c, then $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(c)$.

Let $\boxed{c} \in \mathbb{R} \setminus \mathbb{Q}$ and let $x_n \in \mathbb{R} \setminus \mathbb{Q}$ such that $x_n \to c$ as $n \to \infty$.
Then

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0 = f(c)$$

Thus ② holds in the theorem before.

Now let $x_n \in \mathbb{Q}_n$ such that $x_n \to c$ as $n \to \infty$.

$x_n = \frac{p_n}{q_n}$ in lowest terms where $p_n, q_n \in \mathbb{N}$

Then $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f\left(\frac{p_n}{q_n}\right) = \lim_{n\to\infty} \frac{1}{q_n}$

To show the above limit is zero I need to show $q_n \to \infty$.

For contradiction suppose $q_n \not\to \infty$ then there is a subsequence q_{n_k} that is bounded.

By the Bolzano Weierstrass theorem q_{n_k} has a convergent subsequence $q_{n_{k_j}} \to m$ for some $m \in \mathbb{R}$.

Since $q_{n_{k_j}}$ is a sequence of integers the only it can converge is if it is eventually constant. Thus there is J_0 so that

$$q_{n_{k_j}} = l \quad \text{for all} \quad j \geq J_0.$$

But then $\lim_{j\to\infty} q_{n_{k_j}} = l = m$ so $m \in \mathbb{N}$.

Recall $x_n \to c$ so $\frac{p_{n_{k_j}}}{q_{n_{k_j}}} \to c$.

Moreover for $j \geq J_0$ we have $\frac{p_{n_{k_j}}}{q_{n_{k_j}}} = \frac{p_{n_{k_j}}}{l}$.

Thus $p_{n_{k_j}} \to cl$ as $j \to \infty$.

Note again $p_{n_{k_j}}$ is a sequence of integers so it is eventually constant. Thus the limit $cl \in \mathbb{N}$.

By assumption $c \in \mathbb{R}\setminus\mathbb{Q}$ but $l \in \mathbb{N}$ implies $cl \in \mathbb{R}\setminus\mathbb{Q}$ contradicting $cl \in \mathbb{N}$. Therefore $q_n \to \infty$ as $n \to \infty$

and

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f\left(\frac{p_n}{q_n}\right) = \lim_{n\to\infty} \frac{1}{q_n} = 0$$

Combining ① and ② we obtain that f is cont on $\mathbb{R}\setminus\mathbb{Q}$.

Claim f is discontinuous at every rational number.

$$f(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbb{R}\setminus\mathbb{Q} \\ \frac{1}{n} & \text{if} \quad x \in \mathbb{Q} \text{ where } x = \frac{m}{n} \text{ in lowest terms with } m, n \in \mathbb{N}. \end{cases}$$

Let $c \in \mathbb{Q}$ then $c = \frac{m}{n}$ in lowest terms so $f(c) = \frac{1}{n} \neq 0$

On the other hand let $x_n \in \mathbb{R}\setminus\mathbb{Q}$ with $x_n \to c$.

Then $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0 \neq f(c)$

Thus $f(x_n) \not\to f(c)$ so f discontinuous on the rationals.

For next time look at

Proposition 4.8 If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then f is bounded on $[a, b]$.

- **Theorem 4.2** If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then f has an absolute maximum and an absolute minimum on $[a, b]$.

Theorem 4.3 (Intermediate Value Theorem) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Assume that $f(a) \neq f(b)$. Then, for any k between $f(a)$ and $f(b)$, there is a c in $[a, b]$ such that $f(c) = k$.

Corollary 4.3 The continuous image of a closed interval is again a closed interval (where we allow $\{c\}$ to be the closed interval $[c, c]$).

Now we have enough theoretical foundations to prove these results from the calculus.