Theorem 3.3 Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the closed interval $[a,b]$. Let x be in $\mathbb{R}$ with $x_n \to x$. Then x is in $[a,b]$.

Proposition 3.2 A convergent sequence is bounded.

Theorem 3.10 (Bolzano-Weierstrass Theorem for sequences) A bounded sequence in $\mathbb{R}$ has a convergent subsequence (that is, a subsequence that converges to a real number).

Proposition 4.8 If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then f is bounded on $[a,b]$.

Proof. Prop 4.8,

For contradiction, suppose f were not bounded, Then there are $x_n \in [a,b]$ such that $|f(x_n)| \geq n$.

By the Bolzano-Weierstrass theorem there is a convergent subsequence x_{n_k} and $x \in \mathbb{R}$ such that $x_{n_k} \to x$ as $k \to \infty$.

By Theorem 3.3 since $x_{n_k} \in [a,b]$ then $x \in [a,b]$.

By hypothesis f is continuous on $[a,b]$ and so continuous at x.

Thus $f(x_{n_k}) \to f(x)$ as $k \to \infty$.

Since a convergent sequence is bounded by Prop 3.2, then $f(x_{n_k})$ is bounded, this contradicts

$|f(x_{n_k})| \geq n_k \geq k$.

Theorem 4.2 If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then f has an absolute maximum and an absolute minimum on $[a, b]$.

By Prop 4.8 the set $A = \{ f(x) : x \in [a,b] \}$ is bounded.

Thus $\alpha = \inf A \in \mathbb{R}$ and $\beta = \sup A \in \mathbb{R}$.

Claim max A exists and min A exists.

Consider the supremum β. Thus, β is the least upper bound of A.

For each $n \in \mathbb{N}$ then $\beta - \frac{1}{n}$ is smaller so not an upper bound.

Therefore there is $f(x_n) \in A$ such that $f(x_n) > \beta - \frac{1}{n}$.

Since $x_n \in [a,b]$ are bounded, there is a convergent subsequence $x_{n_k} \to x$ where $x \in [a,b]$ (see Theorem 3.3).

Since f is continuous then $f(x_{n_k}) \to f(x)$ as $k \to \infty$.

Claim $f(x) = \beta$. By construction $\beta \geq f(x_{n_k}) > \beta - \frac{1}{n_k}$

Therefore
$$|f(x_{n_k}) - \beta| \leq \frac{1}{n_k} \to 0 \text{ as } k \to \infty,$$

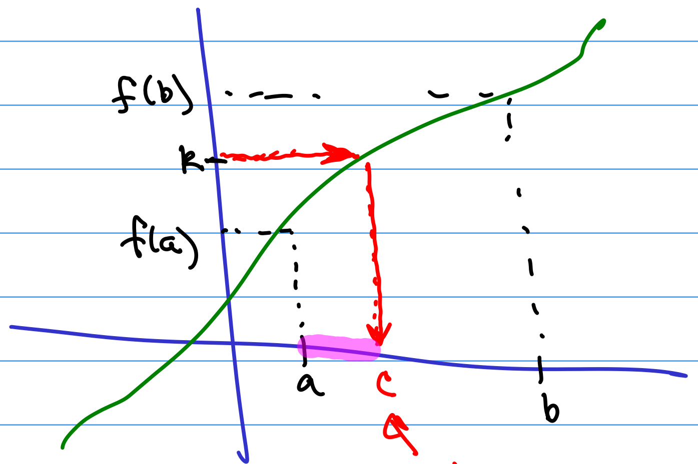
Consequently $f(x_{n_k}) \to \beta$.

Since limits are unique, then $f(x) = \beta$.

This means $\beta \in A$ so the max exists.

Theorem 4.3 (Intermediate Value Theorem) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Assume that $f(a) \neq f(b)$. Then, for any k between $f(a)$ and $f(b)$, there is a c in $[a, b]$ such that $f(c) = k$.

Proof:



Let $S = \{ x \in [a,b] : f(x) < k \}$ define $c = \sup S$.

Note, since S is bounded, then $c \in \mathbb{R}$ by the completeness axiom.

Now, one needs to show $f(c) = k$ using the continuity of the function f. Next time...