Continuity of $f: D \to \mathbb{R}$ means

$\forall x \in D, \forall \varepsilon > 0 \ \exists \delta > 0$ st. $y \in D$ and $|x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

Uniform continuity of $f: D \to \mathbb{R}$ means.

$\forall \varepsilon > 0 \ \exists \delta > 0$ st. $x, y \in D$ and $|x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

Theorem 4.4 A continuous function on a closed interval is uniformly continuous there.

If $f: [a,b] \to \mathbb{R}$ is continuous then it's uniformly continuous.

Suppose f were not uniformly continuous. Then

not $\left(\forall \varepsilon > 0 \ \exists \delta > 0 \text{ st. } \forall x, y \in D \text{ and } |x-y| < \delta \text{ implies } |f(x) - f(y)| < \varepsilon \right)$

equivalently

$\exists \varepsilon > 0$ s.t. $\forall \delta > 0 \ \exists \ x, y \in D$ with $|x-y| < \delta$ s.t. $|f(x) - f(y)| \geq \varepsilon$

Let $n \in \mathbb{N}$ and set $\delta = \frac{1}{n}$.

Then $\exists \ x_n, y_n \in D$ with $\boxed{|x_n - y_n| < \frac{1}{n}}$ s.t $|f(x_n) - f(y_n)| \geq \varepsilon$.

Recall $D = [a,b]$ by hypothesis. Therefore x_n and y_n are bounded. By the Bolzano-Weierstrass theorem there exists a convergent subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$. Let $x \in \mathbb{R}$ be the limit so $x_{n_k} \to x$. By some theorem (look this up) $x \in [a,b]$ since the x_{n_k}'s were in $[a,b]$.

Since f is continuous at x by hypothesis, then for $\varepsilon_1 = \varepsilon/2$

(*) there is $\delta_1 > 0$ such that $y \in D$ and $|x-y| < \delta_1$ implies $|f(x) - f(y)| < \varepsilon_1$.

Since $x_{n_k} \to x$ there is $k_0 \in \mathbb{N}$ such that $k \geq k_0$ implies

$$|x_{n_k} - x| < \delta_1/2 < \delta_1$$

Thus by (*) $\quad |f(x_{n_k}) - f(x)| < \varepsilon_1$

Estimate

$$|y_{n_k} - x| = |y_{n_k} - x_{n_k} + x_{n_k} - x| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - x|$$
$$< \frac{1}{n_k} + \delta_1/2$$

Then for $k_1 \in \mathbb{N}$ large enough $\frac{1}{n_k} < \frac{\delta_1}{2}$ for $k \geq k_1$ so

$$|y_{n_k} - x| \leq \frac{\delta_1}{2} + \frac{\delta_1}{2} = \delta_1 \quad \text{for } k \geq \max(k_0, k_1)$$

Thus by (*) $\quad |f(y_{n_k}) - f(x)| < \varepsilon_1$.

Therefore

$$\varepsilon \leq |f(x_{n_k}) - f(y_{n_k})| \leq |f(x_{n_k}) - f(x)| + |f(y_{n_k}) - f(x)|$$
$$< \varepsilon_1 + \varepsilon_1 = 2\varepsilon_1 = \varepsilon$$

which is a contradiction.

Theorem 4.5 Let $f : D \to \mathbb{R}$ be uniformly continuous on D. If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in D, then $(f(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$.

Recall $(x_n)_{n\in\mathbb{N}}$ is Cauchy means

$$\forall \varepsilon > 0\ \exists\, n_0 \in \mathbb{N} \text{ s.t. } m,n \geq n_0 \text{ implies } |x_n - x_m| < \varepsilon.$$

Proof: Let $\varepsilon > 0$.

Since f is uniformly continuous

(**) $\exists \delta > 0$ s.t. $x, y \in D$ and $|x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

Choose $\varepsilon_1 = \delta_1$. Then since x_n is Cauchy

$$\exists\, n_0 \in \mathbb{N} \text{ s.t. } m,n \geq n_0 \text{ implies } |x_n - x_m| < \varepsilon_1 = \delta.$$

Therefore, by (**) it follows that

$$|f(x_n) - f(x_m)| < \varepsilon \text{ for } m,n \geq n_0$$

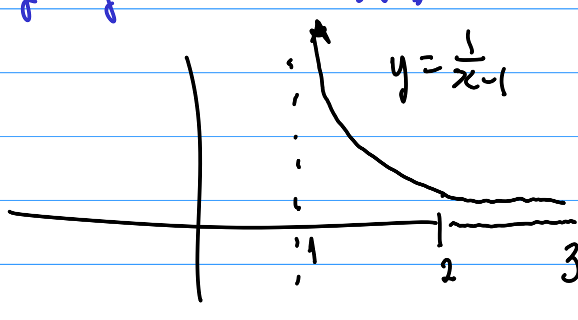
Thus $(f(x_n))_{n\in\mathbb{N}}$ is Cauchy.

Remark: The above proof just involved fitting one definition into the other.

Example: Let $f: (1, 3) \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x-1}$.

Claim that f is not uniformly continuous...

If it were then $(x_n)_{n\in\mathbb{N}}$ Cauchy would imply that $(f(x_n))_{n\in\mathbb{N}}$ also be Cauchy by Theorem 4.5.



Let $x_n = 1 + \frac{1}{n}$. Then $x_n \to 1$ so $(x_n)_{n\in\mathbb{N}}$ is convergent and therefore Cauchy.

However

$$f(x_n) = \frac{1}{x_n - 1} = \frac{1}{1+\frac{1}{n}-1} = n.$$

and this sequence is not even bounded, so not Cauchy.

Theorem 4.6 Let D be a bounded subset of $\mathbb{R}$ and let $f : D \to \mathbb{R}$. Assume that whenever $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in D, $(f(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence. Then f is uniformly continuous on D.

... like a converse to Theorem 4.5

Theorem 4.7 If f is uniformly continuous on (a, b), then f has a continuous extension to $[a, b]$.

How to prove Theorem 4.6? Proof by contradiction?

Proof Suppose f is not uniformly continuous on D. From the proof of Theorem 4.4 (with $[a, b]$ replaced by D), there exist sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, both in D, and an $\varepsilon > 0$ such that $|x_n - y_n| < 1/n$ and $|f(x_n) - f(y_n)| \geq \varepsilon$ for each n in $\mathbb{N}$.

Since D is bounded, by Theorem 3.10, there is a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n\in\mathbb{N}}$ and an L in $\mathbb{R}$ such that $x_{n_k} \underset{k}{\to} L$. Note that L may or may not be in D. Also, as in the proof of Theorem 4.4, $y_{n_k} \underset{k}{\to} L$.

Consider the sequence $(z_n)_{n\in\mathbb{N}} = (x_{n_1}, y_{n_1}, x_{n_2}, y_{n_2}, x_{n_3}, y_{n_3}, \ldots)$. Then $z_n \underset{n}{\to} L$ (Exercise 4 in Section 3.3) and hence $(z_n)_{n\in\mathbb{N}}$ is Cauchy. For each k in $\mathbb{N}$,

$$|f(z_{2k-1}) - f(z_{2k})| = \left|f(x_{n_k}) - f(y_{n_k})\right| \geq \varepsilon,$$

and so $(f(z_n))_{n\in\mathbb{N}}$ is not Cauchy. ∎