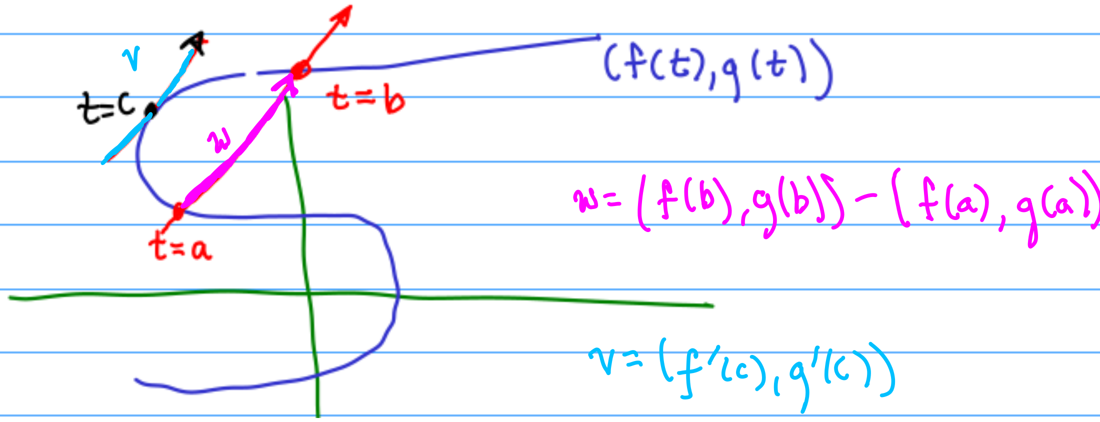
Generalized Mean Value theorem next time

$f:[a,b]\to\mathbb{R}$ $\quad g:[a,b]\to\mathbb{R}$ continuous and differentiable on (a,b).



What do I want ... The vectors point in the same direction

$$w \doteq \alpha v$$

$$(f(b),g(b)) - (f(a),g(a)) = \alpha\,(f'(c),g'(c))$$

Thus

$$f(b)-f(a) = \alpha f'(c)$$

$$\alpha g'(c) = g(b)-g(a)$$

mult

$$\alpha\,(f(b)-f(a))\,g'(c) = \alpha\, f'(c)\,(g(b)-g(a))$$

Eliminate the α.

(*) $$(f(b)-f(a))\,g'(c) = f'(c)\,(g(b)-g(a))$$

Goal: Show there is a c such that the above holds...

Note that if (*) holds one can reverse the above steps to obtain the picture with two parallel lines... the only difficulty might be if $f(b)=f(a)$ ... but then we know for sure that $g(b)\neq g(a)$ ... alternatively if $g(b)=g(a)$ then $f(a)\neq f(b)$

$$(f(b)-f(a))g'(c) = f'(c)(g(b)-g(a))$$

Proposition 5.2 Suppose that f has a local maximum or a local minimum at an interior point c of I. If f is differentiable at c, then $f'(c) = 0$.

want to use Prop 5.2 to show that c exists without having to solve for it ...

$$h(c) = (f(b)-f(a))g'(c) - f'(c)(g(b)-g(a))$$

want $h(c) = 0$

$$H(x) = (f(b)-f(a))g(x) - f(x)(g(b)-g(a))$$

Thus $H'(x) = h(x)$.

Need to show $H'(x) = 0$ for some x. By Prop 5.2 it is enough to show H has a max or min at an iterior point

$$H(a) = (f(b)-f(a))g(a) - f(a)(g(b)-g(a))$$
$$= f(b)g(a) - f(a)g(b)$$

$$H(b) = (f(b)-f(a))g(b) - f(b)(g(b)-g(a))$$
$$= -f(a)g(b) + f(b)g(a)$$

Thus $H(a) = H(b)$.

If the maximum of H wa at the enpoint then

$$H(x) \le H(a) \text{ for all } x \in [a,b]$$

If the minimum are also at an endpoint then

$$H(x) \geq H(a) \text{ for all } x \in [a,b]$$

In this case $H(x)$ is constant so $H(c)$ is also a minimum for any interior point and $H'(c)=0$.

Otherwise there is an extremum (either min, max or both) at an interior point and again at that extremum we have $H'(c)=0$.

In other words.

$$h(c) = (f(b)-f(a))g'(c) - f'(c)(g(b)-g(a)) = 0$$

so we're done. I.e.,

$$(f(b)-f(a))g'(c) = f'(c)(g(b)-g(a))$$

Taylors Theorem:

Given a n-times differentiable function and a point $x_0 \in I$ Taylors polynomial P_n of degree n.

$$P_n(x) = \sum_{k=0}^{n} a_k x^k$$

n+1 coefficients

Such that

$$\left.\begin{aligned} f(x_0) &= P_n(x_0) \\ f'(x_0) &= P_n'(x) \\ &\vdots \\ f^{(n)}(x_0) &= P_n^{(n)}(x) \end{aligned}\right\}$$

n+1 equations...

linear equations in a_k

Can solve for the coefficients using the equations

$$P_n(x) = \sum_{k=0}^{n} \frac{f^k(x_0)}{k!}(x-x_0)^k$$

Idea is approximate f by the polynomial

$$f(x) \approx P_n(x)$$

What is the error in the approximation

$$R_n(x) = f(x) - P_n(x)$$

Taylor's theorem: Given all of the above, assume one more thing, that f has $n+1$ derivatives. Then there exists a c between x and x_0 such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

Proof: Try to solve for c:

$$\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1} = f(x) - P_n(x)$$

$$f^{(n+1)}(c) = \boxed{\frac{f(x)-P_n(x)}{(x-x_0)^{n+1}}}(n+1)! = M(n+1)!$$

constant that doesn't depend on t

Define

$$g(t) = f(t) - P_n(t) - M(t-x_0)^{n+1}$$

supposed to be $g^{(n+1)}(c)$

$$g^{(n+1)}(t) = f^{(n+1)}(t) - P_n^{(n+1)}(t) - M(n+1)!$$

Note

all zero

$$g(x_0) = \cancel{f(x_0)} - \cancel{P_n(x_0)} - M(\cancel{x_0 - x_0})^{n+1} = 0$$

$$g'(x_0) = \cancel{f'(x_0)} - P_n'(x_0) - M(n+1)(\cancel{x_0 - x_0})^n = 0$$

$$\vdots$$

$$g^{(n)}(x_0) = \cancel{f^{(n)}(x_0)} - \cancel{P_n^{(n)}(x_0)} - M(n+1)(n)\cdots 2(\cancel{x_0 - x_0})^1 = 0$$

recall

$$f(x_0) = P_n(x_0)$$

$$f'(x_0) = P_n'(x)$$

$$\vdots$$

$$f^{(n)}(x_0) = P_n^{(n)}(x)$$

$$g^{(n+1)}(x_0) = f^{(n+1)}(x_0) - M(n+1)!$$

$$= f^{(n+1)}(x_0) - \boxed{\frac{f(x) - P_n(x)}{(x - x_0)^{n+1}}}(n+1)!$$ maybe not 0.

$$g(t) = f(t) - P_n(t) - M(t - x_0)^{n+1}$$

$$g(x) = f(x) - P_n(x) - M(x - x_0)^{n+1}$$

$$= \cancel{f(x)} - \cancel{P_n(x)} - \frac{\cancel{f(x)} - \cancel{P_n(x)}}{\cancel{(x - x_0)^{n+1}}}\cancel{(x - x_0)^{n+1}} = 0$$

Thus $g(x) = 0$ and $g(x_0) = 0$ so there is a local extremum c_1 between x and x_0 such that $g'(c_1) = 0$.

Now $g'(c_1) = 0$ and $g'(x_0) = 0$ so there is a local extremum

c_2 between c_1 and x_0 such that $g''(c_2)=0$.

Then $g''(c_2)=0$ and $g''(x_0)=0$ so there is a local extremum

c_3 between c_2 and x_0 such that $g^{(3)}(c_3)=0$.

$\vdots$

Thus $g^{(n)}(c_n)=0$ and $g^{(n)}(x_0)=0$ so there is a local extremum

c between c_n and x_0 such that $g^{(n+1)}(c)=0$.