Theorem 4.7 If f is uniformly continuous on (a, b), then f has a continuous extension to $[a, b]$.

Theorem 4.2 If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then f has an absolute maximum and an absolute minimum on $[a, b]$.

Theorem 6.2 A bounded function f is in $\mathcal{R}[a, b]$ if and only if for every $\varepsilon > 0$ there is a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \varepsilon$.

Theorem 6.7 If f is continuous on $[a, b]$, then f is in $\mathcal{R}[a, b]$.

Proof Use Theorem 6.2. Let $\varepsilon > 0$

Since f is continuous on $[a,b]$ then it is uniformly continuous. (Theorem 4.7)

Thus for $\varepsilon_1 = \boxed{\varepsilon / b-a} > 0$ there exists $\delta_1 > 0$ such that $x', x'' \in [a,b]$ and $|x' - x''| < \delta_1$ imply $|f(x') - f(x'')| < \varepsilon_1$

Let $P \in \mathcal{P}[a,b]$ with $\|P\| < \delta_1$. Then f has a max and min on $[x_{i-1}, x_i]$. (Theorem 4.2).

$M_i = \sup\{f(t) : t \in [x_{i-1}, x_i]\} = f(\alpha_i)$ for some $\alpha_i \in [x_{i-1}, x_i]$

$m_i = \inf\{f(t) : t \in [x_{i-1}, x_i]\} = f(\beta_i)$ for some $\beta_i \in [x_{i-1}, x_i]$

Thus, since $\alpha_i, \beta_i \in [x_{i-1}, x_i]$ and $\Delta x_i < \delta_1$ then $|\alpha_i - \beta_i| < \delta_1$

$$U(P,f) - L(P,f) = \sum_{i=1}^{n} M_i \Delta x_i - \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} (M_i - m_i) \Delta x_i$$

$$= \sum_{i=1}^{n} (f(\alpha_i) - f(\beta_i)) \Delta x_i \leq \sum_{i=1}^{n} |f(\alpha_i) - f(\beta_i)| \Delta x_i$$

$$< \sum_{i=1}^{n} \varepsilon_1 \Delta x_i = \varepsilon_1 \sum_{i=1}^{n} \Delta x_i = \varepsilon_1 (b-a) = \varepsilon$$

Proposition 6.4 Let f and g be in $\mathcal{R}[a,b]$, with $f(x) \le g(x)$ for all x in $[a,b]$. Then $\int_a^b f \le \int_a^b g$.

Theorem 4.3 (Intermediate Value Theorem) Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$. Assume that $f(a) \ne f(b)$. Then, for any k between $f(a)$ and $f(b)$, there is a c in $[a,b]$ such that $f(c) = k$.

Theorem 6.8 (First Mean Value Theorem) If f is continuous on $[a,b]$, then there is a c in $[a,b]$ such that $\int_a^b f = f(c)(b-a)$.

Let $M = \sup\{f(t) : t \in [a,b]\} = f(\alpha)$ for some $\alpha \in [a,b]$

$m = \inf\{f(t) : t \in [a,b]\} = f(\beta)$ for some $\beta \in [a,b]$

(theorem 4.2)

Thus $m \le f(x) \le M$ for $x \in [a,b]$

by prop 6.4 we can integrate the inequality to get.

$\int_a^b m \le \int_a^b f \le \int_a^b M$ $(b-a)m \le \int_a^b f \le (b-a)M$

const

$$f(\beta) = m \le \underbrace{\frac{1}{b-a}\int_a^b f}_{k} \le M = f(\alpha)$$

Since $k = \frac{1}{b-a}\int_a^b f$ is between $f(\beta)$ and $f(\alpha)$ then by the intermediate value theorem (Theorem 4.3) there is c between α and β such that $f(c) = k$.

Therefore $f(c) = \frac{1}{b-a}\int_a^b f$

or $\int_a^b f = (b-a)f(c)$.

Note: If $f(x) \leq g(x)$ it's not necessarily true that $f'(x) \leq g'(x)$

True or false. Suppose f, g are differentiable on $[a,b]$ and $f(x) \leq g(x)$ for $x \in [a,b]$.
Then $f'(x) \leq g'(x)$ for $x \in [a,b]$.

FALSE!

Provide a counter example:

$f(x) = -e^{-x}$ $\quad g(x) = e^{-x}$

$f'(x) = e^{-x}$ $\quad g'(x) = -e^{-x}$

Thus $f(x) \leq g(x)$ but $f(x) \geq g(x)$.

$f(x) = \frac{-1}{x}$ $\quad g(x) = 13$ $\quad [a,b] = [1,2]$

$f'(x) = \frac{1}{x^2}$ $\quad g'(x) = 0$

thus $f(x) \leq g(x)$ for $x \in [1,2]$ but $f'(x) > g'(x)$.

Example 3 $\quad f(x) = x^3$ $\quad g(x) = x^2$ $\quad [a,b] = [-2,-1]$

$f'(x) = 3x^2$ $\quad g'(x) = 2x$

Thus $f(x) \leq g(x)$ and $f'(x) \geq g'(x)$.

$f(x) = \sin x$ $\quad g(x) = 13$

Proposition 6.3 Let f and g be in $\mathcal{R}[a,b]$ and let c be in $\mathbb{R}$. Then $f \pm g$ and cf are in $\mathcal{R}[a,b]$, and

$$\int_a^b (f \pm g) = \int_a^b f \pm \int_a^b g$$

Theorem 6.2 A bounded function f is in $\mathcal{R}[a,b]$ if and only if for every $\varepsilon > 0$ there is a partition P of $[a,b]$ such that $U(P,f) - L(P,f) < \varepsilon$.

Theorem 6.9 If f is monotone on $[a,b]$, then f is in $\mathcal{R}[a,b]$.

Proof: Without loss of generality suppose f is monotone increasing. Otherwise consider $-f$ in its place. Then Prop 6.3 implies

$$\int_a^b -f = \int_a^b 0 - f = \int_a^b 0 - \int_a^b f = -\int_a^b f$$

So, $$\int_a^b f = -\int_a^b -f.$$ Thus $-f \in \mathcal{R}[a,b]$ implies $f \in \mathcal{R}[a,b]$

If $f(a) = f(b)$. Then f is constant and so $f \in \mathcal{R}[a,b]$.

Now consider the case $f(a) < f(b)$.

Let $\varepsilon > 0$. Let $P \in \mathcal{P}[a,b]$ with $\|P\| < \delta$ where $\delta = \boxed{\varepsilon/(f(b)-f(a))}$

Then

$$M_i = \sup\{f(t) : t \in [x_{i-1}, x_i]\} = f(x_i) \qquad m_i = \inf\{f(t) : t \in [x_{i-1}, x_i]\} = f(x_{i-1})$$

Estimate

$$U(P,f) - L(P,f) = \sum_{i=1}^n (M_i - m_i)\Delta x_i = \sum_{i=1}^n \left(f(x_i) - f(x_{i-1})\right) \Delta x_i$$

$$\leq \delta \sum_{i=1}^n \left(f(x_i) - f(x_{i-1})\right) = \delta\left(\cancel{f(x_1)} - f(x_0) + \cancel{f(x_2)} - \cancel{f(x_1)} + \cancel{\cdots} + f(x_n) - \cancel{f(x_{n-1})}\right)$$

$$= \delta\left(f(x_n) - f(x_0)\right) = \delta\left(f(b) - f(a)\right) = \varepsilon$$