Completeness axiom

But first: The extended real numbers

$$\mathbb{R}^{\#} = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$$

ordering $-\infty < x < \infty$ for all $x \in \mathbb{R}$.

also $-\infty < \infty$.

Note: for any $u, v \in \mathbb{R}^{\#}$ then

either $u = v$, $u < v$ or $u > v$

but only one of the above hold.

Definition: Supremum and Infimum of a set.

least upper bound   greatest lower bound.

Let $S \subseteq \mathbb{R}^{\#}$

then $\alpha = \sup S$ means

① α is an upper bound of S

② whenever γ is another upper bound

then $\alpha \leq \gamma$.

also $\beta = \inf S$ means

① β is an lower bound of S

② whenever γ is another lower bound

then $\beta \geq \gamma$.

The inf and sup are related to min and max

Definition of minimum and maximum.

Let $S \subseteq \mathbb{R}^{\#}$ and $s_0, s_1 \in S$ then

$s_0 = \min S$ means $s_0 \leq x$ for all $x \in S$

$s_1 = \max S$ means $s_1 \geq x$ for all $x \in S$.

means s_0 is a lower bound of S, but I don't need to say it's the greatest lower bound because it's assumed that $\boxed{s_0 \in S}$

Theorem: If $\min S$ exists then $\min S = \inf S$.
If $\max S$ exists then $\max S = \sup S$.

Prop. 2.3 in book please read the proof

Completeness Axiom for $\mathbb{R}$ Every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$.

Completeness axiom:

If $S \subseteq \mathbb{R}$ and S is bounded above.
and $S \neq \emptyset$

means there's an upper bound $\gamma \in \mathbb{R}$ such that $s \leq \gamma$ for all $s \in S$.

then $\sup S \in \mathbb{R}$.

Idea: The least upper bound exists and it's a real number.

Corollary: If $S \subseteq \mathbb{R}$ and S is bounded below then $\inf S \in \mathbb{R}$.

Proof: Let $A = \{x \in \mathbb{R} : x \text{ is a lower bound of } S\}$.
(rabbit out of a hat)

Note $A \subseteq \mathbb{R}$ Claim $A \neq \emptyset$. Since S is bounded below there is at least one lower bound in A.

Claim A is bounded above. Let $s \in S$ then since any $\gamma \in A$ is a lower bound of S then $\gamma \leq s$.

Thus $\gamma \le s$ for all $\gamma \in A$ means s is an upper bound of A. So A is bounded above.

By the completeness axiom A has a least upper bound.

Let $\alpha = \sup A$. Claim $\alpha = \inf S$. Why?

Need to show α is the greatest lower bound of S.

Let γ be another lower bound of S. Thus $\gamma \in A$ by definition of A. Since $\alpha = \sup A$ then $\gamma \le \alpha$.

This shows that α is the greatest lower bound except I haven't shown α is a lower bound yet.

Claim α is a lower bound of S.

Let $s \in S$. Then since $\alpha = \sup A$ then

Proof Let S be a nonempty subset of $\mathbb{R}$ that is bounded below. Let

$$A = \{x \in \mathbb{R} : x \text{ is a lower bound of } S\}.$$

Then A is nonempty and A is bounded above by each point in S. By the Completeness Axiom for $\mathbb{R}$, $\alpha = \sup A$ is a real number. We will show that $\alpha = \inf S$. Let s be in S. Then s is an upper bound of A. Since α is the least upper bound of A, $\alpha \le s$. Thus, α is a lower bound of S.

To show that α is the greatest lower bound of S, let γ be a real lower bound of S. (If $\gamma = -\infty$, then clearly $\gamma \le \alpha$.) We need to show that $\gamma \le \alpha$. Since γ is a lower bound of S, γ is an element of A. Since α is an upper bound of A, $\gamma \le \alpha$. ∎

here is the proof from the book. We'll finish it next time.