Completeness axiom: If $S \subseteq \mathbb{R}$, $S \neq \emptyset$ and S is bounded above then $\sup S \in \mathbb{R}$.

Prop. 2.4. If $S \subseteq \mathbb{R}$, $S \neq \emptyset$ and S is bounded below then $\inf S \in \mathbb{R}$.

Proof: Let $A = \{x \in \mathbb{R} : x \text{ is a lower bound of } S\}$.

Clearly $A \subseteq \mathbb{R}$. Since S is bounded below then $A \neq \emptyset$.
Claim A is bounded above. Let $s \in S$. Then if $\gamma \in A$ since it's a lower bound of S it follows that $\gamma \leq s$ forall $\gamma \in A$. Thus s is an upper bound of A.

By the completeness axiom $\sup A \in \mathbb{R}$. Let $\alpha = \sup A$.
Claim $\alpha = \inf S$. Thus need to show α is the greatest lower bound of S.

First: α is a lower bound of S because... Let $s \in S$. Thus s is an upper bound of A. Since α is the least upper bound of A then $\alpha \leq s$.

This shows $\alpha \leq s$ forall $s \in S$. So α is a lower bound.

second: α is the greatest lower bound. Let γ be any other lower bound of S. Then $\gamma \in A$ by definition. Since α is least upper bound of A, then $\gamma \leq \alpha$. This shows α is the greatest of lower bounds of S.

Suppose $S \subseteq \mathbb{R}$, $S \neq \emptyset$ and S is bounded above then $\alpha = \sup S \in \mathbb{R}$.

(*) Let $\varepsilon > 0$ and consider $\alpha - \varepsilon$. Since $\alpha - \varepsilon < \alpha$ and α was the least upper bound, then $\alpha - \varepsilon$ is not an upper bound of S. Thus there is $s_0 \in S$ such that $\alpha - \varepsilon < s_0$.

Suppose $S \subseteq \mathbb{R}$, $S \neq \emptyset$ and S is bounded below then $\beta = \inf S \in \mathbb{R}$.

Let $\varepsilon > 0$ and consider $\beta + \varepsilon$. Since $\beta + \varepsilon > \beta$ and β was the greatest lower bound, then $\beta + \varepsilon$ is not an lower bound of S. Thus there is $s_1 \in S$ such that $\beta + \varepsilon > s_1$.

Recall: $\mathbb{N}$ is well ordered. (Axiom 2.10).

F is well ordered if for every $A \subseteq F$, $A \neq \emptyset$ then A has a least element. I.e. there is $a_0 \in A$ such that $a_0 \leq a$ for all $a \in A$.

Recall: Well ordering of $\mathbb{N}$ implies induction is true. (Theorem 1.3).

Induction true means that if the base case and induction step hold then the statement is true for all $n \in \mathbb{N}$.

Theorem 2.1: $\mathbb{N}$ is unbounded above.

Proof: Suppose not. Then $\mathbb{N}$ is bounded above. Since $\mathbb{N} \subseteq \mathbb{R}$ and $\mathbb{N} \neq \emptyset$ then by the completeness axiom $\alpha = \sup \mathbb{N} \in \mathbb{R}$.

Let $\varepsilon = 1$ then by (*) there is $n_0 \in \mathbb{N}$ such that $\alpha - 1 < n_0$. Since $n_0 \in \mathbb{N}$ implies $n_0 + 1 \in \mathbb{N}$ then $\alpha < n_0 + 1$ implies α is not an upper bound of $\mathbb{N}$. Contradicting that α was the least upper bound.

Archimedean Principle: If $x \in \mathbb{R}$ and $x > 0$ then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.

Proof: Since $\mathbb{N}$ is unbounded above there is $n \in \mathbb{N}$ such that $\frac{1}{x} < n$. Then $\frac{1}{n} < x$.

Theorem 2.2 $\mathbb{Q}$ is dense in $\mathbb{R}$.

What does dense mean?

Definition: $A \subseteq \mathbb{R}$ is dense in $\mathbb{R}$ if between any two real numbers the exists an element of A.

equivalently,

$\forall x, y \in \mathbb{R}$ s.t. $x < y$ then $A \cap (x, y) \neq \emptyset$.