HW6 due Friday, Mar 15

Quiz 3 on Monday Mar 18 over homework

Practice 3.3#1abc, 3.3#3, 3.3#4, 3.4#2, 3.4#3, 3.5#2, 3.5#3

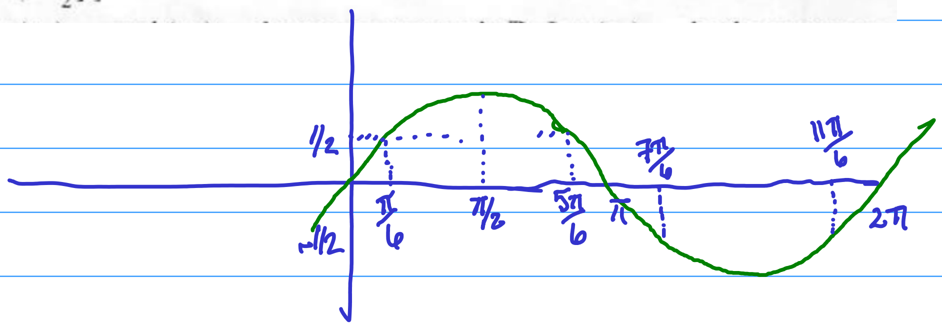
1. (a) Give an example of an unbounded sequence with a convergent subsequence.
   (b) Give an example of an unbounded sequence without a convergent subsequence.
   (c) Can you give an example of a bounded sequence that does not have a convergent subsequence?

(a) let $x_n = \begin{cases} n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$

(b) let $x_n = n$

(c) By the Bolzano-Weierstrass theorem every bounded sequence has a convergent subsequence. Therefore it is not possible to give an example of a bounded sequence that does not have a convergent subsequence.

3. Show that $(\sin n)_{n\in\mathbb{N}}$ does not converge. [*Hint:* Find a subsequence each of whose terms is in $[\frac{1}{2}, 1]$ and another subsequence each of whose terms is in $[-1, -\frac{1}{2}]$.]



Following the hint note that

$\sin\theta \in [1/2, 1]$ when $\theta - 2k\pi \in [\frac{\pi}{6}, \frac{5\pi}{6}]$ for some $k \in \mathbb{Z}$.

$\sin\theta \in [-1, -1/2]$ when $\theta - 2k\pi \in [\frac{7\pi}{6}, \frac{11\pi}{6}]$ for some $k \in \mathbb{Z}$.

Since the lengths of these intervals is $\frac{4\pi}{6} > 1$. Then for each $k \in \mathbb{N}$ there is an $n_k \in \mathbb{N}$ such that

$$n_k \in \left[\frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi\right]$$

and an $m_k \in \mathbb{N}$ such that

$$m_k \in \left[\frac{7\pi}{6} + 2k\pi, \frac{11\pi}{6} + 2k\pi\right].$$

Since the intervals are disjoint, then $n_1 < n_2 < \dots$ and also $m_1 < m_2 < \dots$. Therefore $(\sin n_k)_{k \in \mathbb{N}}$ and $(\sin m_k)_{k \in \mathbb{N}}$ are subsequences of $(\sin n)_{n \in \mathbb{N}}$.

Suppose for contradiction that $(\sin n)_{n \in \mathbb{N}}$ converged. Then there would be $x \in \mathbb{R}$ such that

$$\lim_{n \to \infty} \sin n = x.$$

Since all subsequences converge, we would further have that

$$\lim_{k \to \infty} \sin n_k = x \quad \text{and} \quad \lim_{k \to \infty} \sin m_k = x$$

However since

$$\sin n_k \in [1/2, 1] \text{ and } \sin m_k \in [-1, -1/2] \quad \text{for all } k$$

That would imply $x \in [1/2, 1]$ and also $x \in [-1, -1/2]$. That is a contradiction. Therefore $(\sin n)_{n \in \mathbb{N}}$ does not converge.

4. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be two sequences in $\mathbb{R}$. Let $(z_n)_{n\in\mathbb{N}}$ be the sequence $(x_1, y_1, x_2, y_2, x_3, y_3, \ldots)$. Show that $(z_n)_{n\in\mathbb{N}}$ has a limit in $\mathbb{R}$ if and only if both $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ have the same limit in $\mathbb{R}$.

" $\Rightarrow$ " Suppose $(z_n)_{n\in\mathbb{N}}$ has a limit. Then by Theorem 3.6 every subsequence also converges to the same limit. Thus $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ have the same limit.

" $\Leftarrow$ " Suppose $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ have the same limit. Then there is $x\in\mathbb{R}$ such that

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = x.$$

Let $\varepsilon>0$. By definition of limit there is $n_1\in\mathbb{N}$ such that $n\geq n_1$ implies $|x_n - x| \leq \varepsilon$.

Similarly there is $n_2\in\mathbb{N}$ such that $n\geq n_2$ implies $|y_n - x| < \varepsilon$.

Let $n_0 = 2\max(n_1, n_2)$ and suppose $n\geq n_0$

If $n=2k-1$ is odd then $z_n = x_k$ and $2k = n+1 \geq 2n_1+1 \geq 2n_1$ implies $k\geq n_1$. Thus $|z_n - x| = |x_k - x| < \varepsilon$.

If $n=2k$ is even then $z_n = y_k$ and $2k = n \geq 2n_2$ implies that $k\geq n_2$. Thus $|z_n - x| = |y_k - x| < \varepsilon$.

In both cases $n\geq n_0$ implies $|z_n - x| < \varepsilon$. Therefore the sequence $(z_n)_{n\in\mathbb{N}}$ converges.

2. Let $x_1 = 3$ and $x_{n+1} = 2 - (1/x_n)$ for all n in $\mathbb{N}$. Show that $(x_n)_{n\in\mathbb{N}}$ converges, and find the limit.

Claim that x_n is monotone and bounded. First note

$x_1 = 3$.

$x_2 = 2 - \frac{1}{3} = \frac{6-1}{3} = \frac{5}{3} \leq x_1$

$x_3 = 2 - \frac{3}{5} = \frac{10-3}{5} = \frac{7}{5} \leq x_2$

$x_4 = 2 - \frac{5}{7} = \frac{14-5}{7} = \frac{9}{7} \leq x_3$

To see that 1 is a lower bound procede by induction. Clearly $x_1 \geq 1$ so the base case is satisfied.

Suppose $x_n \geq 1$. Then

$$x_{n+1} = 2 - \frac{1}{x_n} \geq 2 - 1 = 1$$

shows that $x_{n+1} \geq 1$. Therefore $x_n \geq 1$ for all $n \in \mathbb{N}$.

To see that x_n is decreasing consider

$$x_n - x_{n+1} = x_n - \left(2 - \frac{1}{x_n}\right) = x_n - 2 + \frac{1}{x_n} = \frac{x_n(x_n-2)+1}{x_n}$$

$$= \frac{x_n^2 - 2x_n + 1}{x_n} = \frac{(x_n-1)^2}{x_n} \geq 0$$

since $x_n \geq 1 > 0$ and $(x_n-1)^2$ is never negative. Thus $x_n \geq x_{n+1}$ implies $(x_n)_{n\in\mathbb{N}}$ is monotone decreasing.

By the monotone convergence theorem it follows that there is $x \in \mathbb{R}$ such that $x_n \to x$ as $n \to \infty$. The only thing left is to find x.

By definition

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\left(2 - \frac{1}{x_n}\right) = 2 - \lim_{n\to\infty}\frac{1}{x_n} = 2 - \frac{1}{x}$$

since $x_n \geq 1$ implies $x \neq 0$. Therefore

$$x = 2 - \frac{1}{x} \quad \text{or} \quad x^2 - 2x + 1 = 0 \quad \text{or} \quad (x-1)^2 = 0.$$

It follows that $x = 1$.

3. Let $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2 + x_n}$ for all n in $\mathbb{N}$. Show that $(x_n)_{n\in\mathbb{N}}$ converges, and find the limit.

Claim x_n is bounded above by 2. Clearly $x_1 \leq 2$. For induction suppose $x_n \leq 2$. Thus

$$x_{n+1} = \sqrt{2 + x_n} \leq \sqrt{2+2} = \sqrt{4} = 2$$

Therefore $x_{n+1} \leq 2$ thereby completing the induction.

Claim x_n is monotone increasing. Since $2 \geq x_n$ then

$$x_{n+1} = \sqrt{2 + x_n} \geq \sqrt{x_n + x_n} = \sqrt{2x_n} \geq \sqrt{x_n x_n} = x_n$$

Therefore x_n is monotone increasing. By the Monotone convergence theorem there is $x \in \mathbb{R}$ such that $x_n \to x$ as $n \to \infty$. It follows that

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{2+x_n} = \sqrt{2+x}$$

Consequently $x = \sqrt{2+x}$ or $x^2 = 2+x$. Thus

$$x^2 - x - 2 = \left(x - \tfrac{1}{2}\right)^2 - \tfrac{1}{4} - 2 = \left(x-\tfrac{1}{2}\right)^2 - \tfrac{9}{4} = 0$$

and so

$$x - \frac{1}{2} = \pm\frac{3}{2} \qquad \text{or} \qquad x = 2 \text{ or } -1$$

Since $x_n \geq x_1 = \sqrt{2}$ then $x = 2$ and $\lim_{n\to\infty} x_n = 2$.

2. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence of integers. Show that $(x_n)_{n\in\mathbb{N}}$ has a subsequence that eventually is constant.

Since $(x_n)_{n\in\mathbb{N}}$ is bounded the Bolzano-Weierstrass theorem implies there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that converges.

Therefore, $\lim_{k\to\infty} x_{n_k} = x$ for some $x \in \mathbb{R}$.

Let $\varepsilon = \frac{1}{4}$. Then there is $n_0 \in \mathbb{N}$ such that $k \geq n_0$ implies $|x_{n_k} - x| \leq \frac{1}{4}$. In other words $x_{n_k} \in [x - \frac{1}{4}, x + \frac{1}{4}]$.

Since the length of this interval is $\frac{1}{2}$ there is at most one integer in it. Since $x_{n_k} \in \mathbb{Z}$ there is at least one integer in this interval. Thus

$$[x - \tfrac{1}{4}, x + \tfrac{1}{4}] \cap \mathbb{Z} = \{p\}$$

for some $p \in \mathbb{Z}$. It follows that $x_{n_k} = p$ for $k \geq n_0$ or that the subsequence $(x_{n_k})_{k\in\mathbb{N}}$ is eventually constant.

3. Show that a bounded sequence in $\mathbb{R}$ that does not converge has more than one subsequential limit. That is, show that a nonconvergent bounded sequence has two subsequences each with a different limit.

We prove the contrapositive if every convergent subsequence of a bounded sequence has the same limit, then the bounded sequence must also converge.

Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence and suppose there is $x\in\mathbb{R}$ such that every convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ satisfies $x_{n_k}\to x$ as $k\to\infty$.

Claim that $x_n\to x$ as $n\to\infty$. If not, then there would exist $\varepsilon>0$ such that for every $n_0\in\mathbb{N}$ there is an $n\geq n_0$ for which $|x_n-x|\geq\varepsilon$. We consequently obtain a subsequence m_j such that $|x_{m_j}-x|\geq\varepsilon$ for all $j\in\mathbb{N}$.

Let $(x_{m_{j_k}})_{k\in\mathbb{N}}$ be a convergent subsequence of $(x_{m_j})_{j\in\mathbb{N}}$ by Bolzano-Weierstrass theorem. By hypothesis $x_{m_{j_k}}\to x$ as $k\to\infty$. Therefore there is k_0 such that $k\geq k_0$ implies $|x_{m_{j_k}}-x|<\varepsilon$.

This contradicts that $|x_{m_j}-x|\geq\varepsilon$.

Therefore $x_n\to x$ as $n\to\infty$.