2. For each n in $\mathbb{N}$, let

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}.$$

(a) Show that $(x_n)_{n\in\mathbb{N}}$ is not Cauchy.
(b) Show that $\lim_{n\to\infty} |x_{n+1} - x_n| = 0$.

(a) To show $(x_n)_{n\in\mathbb{N}}$ is not Cauchy let $n_0 \in \mathbb{N}$ be arbitrarily large. Then for $n, m \geq n_0$ with $n > m$

$$|x_n - x_m| = \left| \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{m} \frac{1}{k} \right| = \sum_{k=m+1}^{n} \frac{1}{k}$$

Taking $n = 2m$ then yields

$$|x_n - x_m| = \sum_{k=m+1}^{2m} \frac{1}{k} \geq \sum_{k=m+1}^{2m} \frac{1}{2m} = \frac{1}{2}$$

Thus, for $\varepsilon = \frac{1}{2}$ and any $n_0 \in \mathbb{N}$ there exists $n, m \geq n_0$, namely take $n = 2m$, such that $|x_n - x_m| \geq \varepsilon$. This means $(x_n)_{n\in\mathbb{N}}$ is not Cauchy.

(b) To show $\lim_{n\to\infty} |x_{n+1} - x_n| = 0$ estimate as

$$|x_{n+1} - x_n| = \left| \sum_{k=1}^{n+1} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} \right| = \frac{1}{n+1}$$

Since $n+1 \to \infty$ as $n \to \infty$ then $\frac{1}{n+1} \to 0$ as $n \to \infty$.

6. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Let $0 < r < 1$ and suppose that

$$|x_{n+2} - x_{n+1}| \leq r\,|x_{n+1} - x_n| \text{ for } n \geq 1.$$

Show that $(x_n)_{n\in\mathbb{N}}$ is Cauchy. [*Hint:* First show that $|x_{n+2} - x_{n+1}| \leq r^n\,|x_2 - x_1|$; then proceed as in Example 3.23.]

Claim that $|x_{n+2} - x_{n+1}| \leq r^n |x_2 - x_1|$. We procede by induction.
For the base case note that

$$|x_{n+2} - x_{n+1}| \leq r|x_{n+1} - x_n|$$

becomes

$$|x_3 - x_2| \leq r|x_2 - x_1| \quad \text{for } n=1$$

which is exactly what's needed.

For the induction step suppose $|x_{n+2} - x_{n+1}| \leq r^n |x_2 - x_1|$. We need to show that $|x_{n+3} - x_{n+2}| \leq r^{n+1} |x_2 - x_1|$.

Since

$$|x_{n+2} - x_{n+1}| \leq r|x_{n+1} - x_n|$$

becomes

$$|x_{n+3} - x_{n+2}| \leq r\,|x_{n+2} - x_{n+1}|$$

when n is replaced by $n+1$, we estimate using the induction hypothesis as

$$|x_{n+3} - x_{n+2}| \leq r\,|x_{n+2} - x_{n+1}| \leq r\left(r^n|x_2 - x_1|\right) = r^{n+1}|x_2 - x_1|.$$

This completes the induction as proves the claim.

Now use the claim to show that $(x_n)_{n\in\mathbb{N}}$ is Cauchy.

Let $\varepsilon > 0$ and choose $n_0 \geq 2$ such that $|x_2 - x_1| \dfrac{r^{n_0-1}}{1-r} < \varepsilon$.

Note, such an n_0 exists because $0 < r < 1$ implies $r^n \to 0$ as $n \to \infty$.

Thus $n, m \geq n_0$ with $n > m$ implies

$$|x_n - x_m| = |x_n - x_{n-1} + x_{n-1} + \cdots + x_{m+1} - x_m|$$

$$\leq \sum_{k=m}^{n-1} |x_{k+1} - x_k| = \sum_{k=m-1}^{n-2} |x_{k+2} - x_{k+1}|$$

$$\leq \sum_{k=m-1}^{n-2} r^k |x_2 - x_1| = |x_2 - x_1| \sum_{k=m-1}^{n-2} r^k .$$

Sum the geometric series $S = \sum_{k=m-1}^{n-2} r^k$ as

$$\begin{aligned} S &= r^{m-1} + \cancel{r^m} + \cancel{\cdots} + \cancel{r^{n-2}} \\ rS &= \cancel{r^m} + \cancel{r^{m+1}} + \cancel{\cdots} + r^{n-1} \\ \hline (1-r)S &= r^{m-1} - r^{n-1} \end{aligned}$$

Therefore $S = \dfrac{r^{m-1} - r^{n-1}}{1-r}$ and

$$|x_n - x_m| \leq |x_2 - x_1| \frac{r^{m-1} - r^{n-1}}{1-r}$$

$$\leq |x_2 - x_1| \frac{r^{m-1}}{1-r} \leq |x_2 - x_1| \frac{r^{n_0-1}}{1-r} < \varepsilon$$

for all $m, n \geq n_0$. Therefore $(x_n)_{n \in \mathbb{N}}$ is Cauchy.

7. Let $x_1 > 0$ and let $x_{n+1} = 1/(2+x_n)$ for $n \geq 1$.

(a) Use Exercise 6 to show that $(x_n)_{n\in\mathbb{N}}$ is Cauchy.

(b) Find $\lim\limits_{n\to\infty} x_n$.

(a) Claim there is $r \in [0,1)$ such that $|x_{n+2} - x_{n+1}| \leq r|x_{n+1} - x_n|$ for all $n \in \mathbb{N}$.

First note since $x_1 > 0$ that $x_n > 0$ for all $n \in \mathbb{N}$.

Next estimate

$$|x_{n+2} - x_{n+1}| = \left| \frac{1}{2+x_{n+1}} - \frac{1}{2+x_n} \right| = \left| \frac{\cancel{2}+x_n - \cancel{2} - x_{n+1}}{(2+x_{n+1})(2+x_n)} \right|$$

$$= \frac{|x_{n+1} - x_n|}{|2+x_{n+1}|\,|2+x_n|} \leq \frac{|x_{n+1} - x_n|}{|2|\,|2|} = r|x_{n+1} - x_n|$$

where $r = \frac{1}{4}$. Since $\frac{1}{4} \in [0,1)$ the claim is proved. Therefore by the previous problem $(x_n)_{n\in\mathbb{N}}$ is Cauchy.

(b) Since $(x_n)_{n\in\mathbb{N}}$ converges, we know for some $L \in \mathbb{R}$ that

$$\lim_{n\to\infty} x_n = L$$

Consequently

$$L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{2+x_n} = \frac{1}{2+L}$$

implies $L = \frac{1}{2+L}$ or $L^2 + 2L - 1 = 0$. The roots of this quadratic are

$$L = \frac{-2 \pm \sqrt{8}}{2} = -1 \pm \sqrt{2}.$$

Since $x_n > 0$ for all $n \in \mathbb{N}$ we know $L \geq 0$. Consequently, we know since $-1-\sqrt{2} < 0$ that the limit is

$$\lim_{n\to\infty} x_n = -1+\sqrt{2}.$$

1. Use the method of Examples 3.24 and 3.25 to establish the following limits.

(d) $\lim_{n\to\infty} (n - 6\sqrt{n}) = \infty$

Estimate for $n \geq 49$ we have $\sqrt{n} \geq 7$ and

$$n - 6\sqrt{n} = \sqrt{n}(\sqrt{n} - 6) \geq \sqrt{n}(7-6) = \sqrt{n}$$

Let $\alpha > 0$ and choose $n_0 > \max(49, \alpha^2)$. Then for $n \geq n_0$ we have $\sqrt{n} > 7$ and $\sqrt{n} > \alpha$. It follows that

$$n - 6\sqrt{n} \geq \sqrt{n} > \alpha$$

and so $\lim_{n\to\infty} (n - 6\sqrt{n}) = \infty$.

6. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences of positive real numbers and suppose that $x_n/y_n \to L$ where $0 < L < \infty$. Show that $x_n \to \infty$ if and only if $y_n \to \infty$. [*Hint:* Note that, eventually, $\frac{1}{2}L < x_n/y_n < \frac{3}{2}L$.]

Since $L>0$ then taking $\varepsilon = \frac{L}{2}$ yields the neighborhood of L given by $V = (L-\varepsilon, L+\varepsilon) = (\frac{1}{2}L, \frac{3}{2}L)$.

Since $\frac{x_n}{y_n} \to L$ then $\left(\frac{x_n}{y_n}\right)_{n\in\mathbb{N}}$ is eventually in $(\frac{1}{2}L, \frac{3}{2}L)$.

Thus, there is $n_1 \in \mathbb{N}$ such that $\frac{1}{2}L < \frac{x_n}{y_n} < \frac{3}{2}L$ for all $n \geq n_1$.

"$\Rightarrow$" Suppose $x_n \to \infty$. Claim that $y_n \to \infty$. Since x_n and y_n are positive, then

$$\frac{x_n}{y_n} < \frac{3}{2}L \quad \text{implies} \quad y_n > \frac{2}{3L}x_n \quad \text{for } n \geq n_1$$

Let $\alpha > 0$ be arbitrarily large. Since $x_n \to \infty$ there is $n_2 \geq n_1$ such that $n \geq n_2$ implies $x_n > \frac{3}{2}L\alpha$.

It follows for $n \geq n_2$ that

$$y_n > \frac{2}{3L}x_n > \cancel{\frac{2}{3L}}\cancel{\frac{3}{2}L}\alpha = \alpha.$$

Therefore $y_n \to \infty$ as $n \to \infty$.

"$\Leftarrow$" Suppose $y_n \to \infty$. Claim that $x_n \to \infty$. Since x_n and y_n are positive, then

$$\frac{1}{2}L < \frac{x_n}{y_n} \quad \text{implies} \quad x_n > \frac{1}{2}L y_n \quad \text{for} \quad n \geq n_1$$

Let $\alpha > 0$ be arbitrarily large. Since $y_n \to \infty$ there is $n_3 \geq n_1$ such that $n \geq n_3$ implies $y_n > \frac{2}{L}\alpha$.

It follows for $n \geq n_3$ that

$$x_n > \frac{1}{2}L y_n > \cancel{\frac{1}{2}}L \cancel{\frac{2}{L}} \alpha = \alpha.$$

Therefore $x_n \to \infty$ as $n \to \infty$.

1. Find lim sup x_n and lim inf x_n if x_n is given by

(c) $(-1)^n \left(1 + \frac{1}{n}\right)$.

By definition

$$x_n = \begin{cases} 1 + \frac{1}{2k} & \text{if } n = 2k \text{ is even} \\ -1 - \frac{1}{2k-1} & \text{if } n = 2k-1 \text{ is odd} \end{cases}.$$

As $1 + \frac{1}{2k} \to 1$ as $k \to \infty$ and $-1 - \frac{1}{2k-1} \to -1$ as $k \to \infty$ then

$$E = \{x \in \mathbb{R}^\# : x_{n_k} \to x \text{ for some subsequence } (x_{n_k})_{k=1}^{\infty} \text{ of } (x_n)_{n \in \mathbb{N}}\}$$

$$E = \{-1, 1\}$$

Consequently

$$\limsup_{n \to \infty} x_n = \sup E = 1 \quad \text{and} \quad \liminf_{n \to \infty} x_n = \inf E = -1.$$

4. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}$. Show that

$$\liminf a_n + \liminf b_n \leq \liminf (a_n + b_n)$$

whenever the left side is defined.

One can mimic the proof of Proposition 3.8 in the book. Alternatively,

let $A = \{ x \in \mathbb{R}^{\#} : a_{n_k} \to x \text{ for some subsequence } (a_{n_k})_{k=1}^{\infty} \text{ of } (a_n)_{n\in\mathbb{N}} \}$

$B = \{ x \in \mathbb{R}^{\#} : b_{n_k} \to x \text{ for some subsequence } (b_{n_k})_{k=1}^{\infty} \text{ of } (b_n)_{n\in\mathbb{N}} \}$

and

$C = \{ x \in \mathbb{R}^{\#} : a_{n_k} + b_{n_k} \to x \text{ for some subsequence } (a_{n_k} + b_{n_k})_{k=1}^{\infty} \text{ of } (a_n + b_n)_{n\in\mathbb{N}} \}$.

By Exercise 2.2#5 $-\inf(S) = \sup(-S)$ where $-S = \{-s \in S : s \in S\}$.
Remark, since 2.2#5 wasn't assigned, I'll prove it after.

Note further that

$-A = \{ x \in \mathbb{R}^{\#} : -a_{n_k} \to x \text{ for some subsequence } (-a_{n_k})_{k=1}^{\infty} \text{ of } (-a_n)_{n\in\mathbb{N}} \}$

$-B = \{ x \in \mathbb{R}^{\#} : -b_{n_k} \to x \text{ for some subsequence } (-b_{n_k})_{k=1}^{\infty} \text{ of } (-b_n)_{n\in\mathbb{N}} \}$

and

$-C = \{ x \in \mathbb{R}^{\#} : -a_{n_k} - b_{n_k} \to x \text{ for some subsequence } (-a_{n_k} - b_{n_k})_{k=1}^{\infty} \text{ of } (-a_n - b_n)_{n\in\mathbb{N}} \}$.

Therefore

$$\limsup_{n\to\infty} -a_n = \sup(-A) = -\inf(A) = -\liminf_{n\to\infty} a_n$$

$$\limsup_{n\to\infty} -b_n = \sup(-B) = -\inf(B) = -\liminf_{n\to\infty} b_n$$

and

$$\limsup_{n\to\infty} (-a_n - b_n) = \sup(-C) = -\inf(C) = -\liminf_{n\to\infty}(a_n + b_n).$$

Now applying Proposition 3.8 to the sequences $(-a_n)_{n\in\mathbb{N}}$ and $(-b_n)_{n\in\mathbb{N}}$ yield

$$\limsup_{n\to\infty} (-a_n - b_n) \leq \limsup_{n\to\infty} (-a_n) + \limsup_{n\to\infty} (-b_n)$$

or equivalently that

$$-\liminf_{n\to\infty} (a_n + b_n) \leq -\liminf_{n\to\infty} (a_n) - \liminf_{n\to\infty} (b_n)$$

Multiplying by -1 then obtains

$$\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n \leq \liminf_{n\to\infty} (a_n + b_n)$$

which was to be shown.