Math 310 Sample Final Exam Version A

1. Let $f\colon I \to \mathbf{R}$ with $c \in I$. Then f has a local minimum at c if and only if ______.

2. A sequence $(x_n)_{n\in\mathbf{N}}$ in $\mathbf{R}$ is monotone increasing if and only if ______.

3. A function $f\colon X \to Y$ is a bijection if and only if ______.

4. Let $f\colon I \to \mathbf{R}$ with $c \in I$. Then f has a local maximum at c if and only if ______.

5. A set A is countable if and only if ______.

6. A sequence $(x_n)_{n\in\mathbf{N}}$ in $\mathbf{R}$ is monotone decreasing if and only if ______.

7. Let A be a subset of $\mathbf{R}$. Then A is dense in $\mathbf{R}$ if and only if ______.

8. A function $f\colon D \to \mathbf{R}$ is continuous at c if and only if ______.

9. A function $f\colon D \to \mathbf{R}$ is uniformly continuous on D if and only if ______.

10. The limit of f at c is L or $\lim_{x\to c} f(x) = L$ if and only if ______.

11. A sequence $(x_n)_{n\in\mathbf{N}}$ in $\mathbf{R}$ is Cauchy if and only if ______.

12. f has a discontinuity of the first kind at c if and only if ______.

Answers:

(A) $x_n \le x_{n+1}$ for all n in $\mathbf{N}$.

(B) there is a neighborhood U of c such that $f(x) \le f(c)$ for all $x \in U \cap I$.

(C) both $f(c+)$ and $f(c-)$ exist in $\mathbf{R}$ and $f(c+) \ne f(c-)$.

(D) $\forall \epsilon > 0\ \exists n_0 \in \mathbf{N}$ such that if $n \ge n_0$ and $m \ge n_0$ then $|x_n - x_m| < \epsilon$.

(E) for every x and y in $\mathbf{R}$ with $x < y$ one has $A \cap (x, y) \ne \emptyset$.

(F) $x_n \ge x_{n+1}$ for all n in $\mathbf{N}$.

(G) for all $\epsilon > 0$ there is a $\delta > 0$ such that if x and y are in D with $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

(H) there is a neighborhood U of c such that $f(x) \ge f(c)$ for all $x \in U \cap I$.

(I) f is one-to-one and onto Y.

(J) for all $\epsilon > 0$ there is a $\delta > 0$ such that if x is in D and $0 < |x - c| < \delta$ then $|f(x) - L| < \epsilon$.

(K) A is finite or A is countably infinite.

(M) for every neighborhood V of $f(c)$ there is a neighborhood U of c such that if x is in $U \cap D$ then $f(x)$ is in V.

13. $x = \sup S$ if and only if ______.

14. $\limsup_{x\to\infty} x_n$ is defined as ______.

15. $\lim_{n\to\infty} x_n = -\infty$ if and only if ______.

16. $\lim_{n\to\infty} x_n = x$ for $x \in \mathbf{R}$ if and only if ______.

17. $\liminf_{x\to\infty} x_n$ is defined as ______.

18. Let $A \subseteq \mathbf{R}$ and $x \in \mathbf{R}$. Then x is an accumulation point of A if ______.

19. $\lim_{n\to\infty} x_n = \infty$ if and only if ______.

20. $x = \inf S$ if and only if ______.

21. Let $f\colon X \to Y$ and $A \subseteq X$. The direct image $f(A)$ is defined as ______.

22. Let $f\colon X \to Y$ and $A \subseteq Y$. The inverse image $f^{-1}(A)$ is defined as ______.

23. Let $\mathcal{U}$ be a collection of sets. Then $\bigcup\mathcal{U}$ is defined as ______.

24. Let $\mathcal{U}$ be a collection of sets. Then $\bigcap\mathcal{U}$ is defined as ______.

Answers:

(A) for all $\beta < 0$ there exists $n_0 \in \mathbf{N}$ such that if $n \geq n_0$ then $x_n < \beta$.

(B) $\{\, x : x \in A$ for all $A \in \mathcal{U}\,\}$.

(C) $\inf\{x \in \mathbf{R}^{\#} : x_{n_k} \to x$ for some subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n\in\mathbf{N}}\}$.

(D) for all $\epsilon > 0$ there exists $n_0 \in \mathbf{N}$ such that if $n \geq n_0$ then $|x_n - x| < \epsilon$.

(E) $\{\, x \in X : f(x) \in A\,\}$.

(F) $\{\, x : x \in A$ for at least one $A \in \mathcal{U}\,\}$.

(G) for all $s \in S$ then $x \geq s$ and if γ is a upper bound of S then $\gamma \geq x$.

(H) if every neighborhood of x contains a point of A different from x.

(I) for all $\alpha > 0$ there exists $n_0 \in \mathbf{N}$ such that if $n \geq n_0$ then $x_n > \alpha$.

(J) $\{\, f(a) : a \in A\,\}$.

(K) $\sup\{x \in \mathbf{R}^{\#} : x_{n_k} \to x$ for some subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n\in\mathbf{N}}\}$.

(M) for all $s \in S$ then $x \leq s$ and if γ is a lower bound of S then $\gamma \leq x$.

25. Give a precise definition of what it means for a function f to be differentiable at c and the value $f'(c)$ of the derivative.

26. Finish the following statement of Taylor's theorem exactly:

> **Theorem 5.6.** Suppose that $f\colon [a,b] \to \mathbf{R}$, n is a positive integer, $f^{(n)}$ is continuous on $[a,b]$, and $f^{(n)}$ is differentiable on (a,b). For $x \neq x_0$ in $[a,b]$, there is ...

27. Find a bounded sequence x_n and a convergent sequence y_n such that $x_n y_n$ does not converge.

28. Prove one of the following:

Proposition 3.2: A convergent sequence is bounded.

Theorem 3.7: A bounded monotone sequence converges.

29. Prove one of the following:

Theorem 4.2: If $f\colon [a,b] \to \mathbf{R}$ is continuous on $[a,b]$ then f has an absolute maximum and an absolute minimum on $[a,b]$.

Theorem 4.4: A continuous function on a closed interval is uniformly continuous there.

30. Let $A = \{x \in \mathbf{R} : x^2 < 2\}$.

(i) Find $\sup A$.

(ii) Find $\inf A$.

31. Fill in the missing parts of the Bolzano–Weierstrass theorem for sequences exactly:

Theorem 3.10. A ... sequence in $\mathbf{R}$ has a ... subsequence.

32. Give an example of an unbounded sequence with a convergent subsequence.

33. Give an example of $f\colon \mathbf{R} \to \mathbf{R}$ such that f is continuous but not uniformly continous.

34. Give an example of a continuous function $g\colon \mathbf{R} \to \mathbf{R}$ such that $g\big((-1,1)\big)$ is not an open interval.

35. Let $f\colon X \to Y$ and $A \subseteq X$. Give an example showing that $f^{-1}(f(A)) = A$ need not hold if f is not one-to-one.

36. For reference recall the following results:

Theorem 6.2: A bounded function f is in $\mathcal{R}[a,b]$ if and only if for every $\varepsilon > 0$ there is a partition P of $[a,b]$ such that $U(P,f) - L(P,f) < \varepsilon$.

Proposition 6.3: Let f and g be in $\mathcal{R}[a,b]$ and let c be in $\mathbf{R}$. Then $f \pm g$ and cf are in $\mathcal{R}[a,b]$ and

$$\int_a^b (f \pm g) = \int_a^b f \pm \int_a^b g \qquad \text{and} \qquad \int_a^b cf = c\int_a^b f.$$

Also recall the theorems stated in Question 29. Now prove one of the following:

Theorem 6.7: If f is continuous on $[a,b]$, then f is in $\mathcal{R}[a,b]$.

Theorem 6.9: If f is monotone on $[a,b]$, then f is in $\mathcal{R}[a,b]$.

37. [EXTRA CREDIT] Prove the chain rule: Let I and J be intervals or rays in $\mathbf{R}$, let $f\colon I \to J$ and $g\colon J \to \mathbf{R}$, and let c be in I with f differentiable at c and g differentiable at $f(c)$. Then the composite function $g \circ f$ is differentiable at c and $(g \circ f)'(c) = g'(f(c))f'(c)$.