Math 310: Chapter 1.1.

Chapter 1 Proofs, Sets, and Functions

1.1 Proofs

Direct proofs: Example

Proposition 1.1 If $\underbrace{n \text{ is an even integer}}_{p}$, then $\underbrace{n^2 \text{ is an even integer}}_{q}$.

← If p then q

implication

Notation $\mathbb{Z}$ ← set of integers.

If n is even, that means $n = 2k$ for some $k \in \mathbb{Z}$.

Then $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$.

Since $2k^2 \in \mathbb{Z}$ then n^2 is even.

Converse: Given if p then q. The converse is if q then p.

In symbols: $p \Rightarrow q$ The converse is $q \Rightarrow p$

Note the converse has a different truth value that the implication.

Example:

$\boxed{\text{If } f \text{ is differentiable at the point } a, \text{ then } f \text{ is continuous at the point } a.}$

← True Theorem from Calculus

Converse of this statement

~~If f is continuous at the point a then f is differentiable at the point a.~~

Is not a true statement in Calculus.

Biconditional: p if and only if q.

means if p then q and if q then p.

i.e. both the implication and its converse hold.

Mathematical short hand

If p then q	$p \Rightarrow q$
p if and only if q	$p \Leftrightarrow q$
set of positive integers.	$\mathbb{N}$
set of Real numbers	$\mathbb{R}$
set of all integers	$\mathbb{Z}$
for every	$\forall$
There exists	$\exists$
Such that	s.t. or :
$\subseteq$	subset
$\supseteq$	superset

Biconditional: p if and only if q.

$$p \Leftrightarrow q$$

which means $p \Rightarrow q$ and $q \Rightarrow p$

Archimedian Principle.

$$\forall x > 0 \ \exists n \in \mathbb{N} \text{ s.t. } \frac{1}{n} < x.$$

for every positive real number x there exists a positive integer n such that $\frac{1}{n} < x$.

<u>Definition</u> (Unbounded Above). $A \subseteq \mathbb{R}$ is unbounded above

$\forall x > 0 \ \exists \ a \in A$ s.t. $x < a$.

for every positive real number x there exists an element a in A such that $x < a$.

Proposition 1.2 $\overbrace{\mathbb{N} \text{ is unbounded above}}^{p}$ $\boxed{\text{if and only if}}$ $\overbrace{\text{the Archimedean Prin-ciple holds}}^{q}$.

$p \Leftrightarrow q$ mean $p \Rightarrow q$ and $q \Rightarrow p$.

"$\Rightarrow$" Suppose $\mathbb{N}$ is unbounded above. Let $x > 0$. Then $\frac{1}{x} > 0$ and since $\mathbb{N}$ is unbounded above there is $n \in \mathbb{N}$ such that $\frac{1}{x} < n$. Therefore $\frac{1}{n} < x$ and so the Archimedian principle holds.

(archimedian principle)
$\forall x > 0 \ \exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < x$

"$\Leftarrow$" Suppose the Archimedian principle holds. Let $x > 0$. Then $\frac{1}{x} > 0$ and by the archimedian priciple there is $n \in \mathbb{N}$ s.t. $\frac{1}{n} < \frac{1}{x}$. Therefore $x < n$ and we see that $\mathbb{N}$ is unbounded above.

<u>Indirect proofs</u>:

Theorem 1.1 $\sqrt{2}$ is an irrational number.

Theorem 1.2 There are infinitely many primes. (/
in Euclid's *Elements*, Book IX, Proposition 20.)

For next time to illustrate indirect proofs...