Contrapositive

The contrapositive of "if p then q"
is "if not q then not p"

Note these two statements have the same truth values...

In our mathematical shorthand:

$p \Rightarrow q \quad \Leftrightarrow \quad \text{not } q \Rightarrow \text{not } p.$

recall from last time

p q ← If p then q (implication)

Proposition 1.1 If n is an even integer, then n^2 is an even integer.

Notation $\mathbb{Z}$ ← set of integers..

direct proof.

If n is even, that means $n = 2k$ for some $k \in \mathbb{Z}$.
Then $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$.
Since $2k^2 \in \mathbb{Z}$ then n^2 is even.

The contrapositive of Prop 1.1.

If not q then not p.

If not n^2 is an even integer. then not n is an even integer.

If n^2 is odd then n is odd

Indirect proof: To prove $p \Rightarrow q$ assume p and assume not q and then hunt for a contradiction.

Easy example:

Prop 1.4 If $a > 0$ (p) then $-a < 0$ (q).

too easy! but try to follow how indirect proof can be used to support this claim...

Assume p and not q and look for a contradiction.

Assume $a > 0$ and not $-a < 0$. Thus $-a \geq 0$.

Therefore $a > 0$

$-a \geq 0$

implies $a - a > 0 + 0$

or $0 > 0$ ← contradiction

Conclusion: If $a > 0$ then $-a < 0$.

Now use (proof by contradiction) indirect proof to to show that

Theorem 1.1 $\sqrt{2}$ is an irrational number.

Theorem 1.2 There are infinitely many primes. (/
in Euclid's *Elements*, Book IX, Proposition 20.)

Theorem 1.1 $\sqrt{2}$ is an irrational number.

For contradiction suppose $\sqrt{2}$ is rational.

Then $\sqrt{2} = \frac{m}{n}$ where $m \in \mathbb{Z}$, $n \in \mathbb{N}$.

take $\frac{m}{n}$ in lowest terms... that means m and n have no common factors.

By definition $\sqrt{2}$ is something that when squared is 2. Thus, squaring both sides yields

$$2 = \frac{m^2}{n^2} \quad \text{and so} \quad 2n^2 = m^2.$$

Since $m^2 = 2n^2$ then m^2 is even.

☑ Claim m is even. Suppose m were instead odd then $m = 2k+1$ and

$$m^2 = (2k+1)^2 = (2k+1)(2k+1) = 4k^2 + 4k + 1$$

$$= 2(2k^2 + 2k) + 1 \quad \text{is also odd.}$$

This contradicts that m^2 is even.

So m must be even

Since m is even then $m = 2l$ for some $l \in \mathbb{Z}$.

Substitute to obtain.

$$2n^2 = (2l)^2 = 4l^2$$

Thus $n^2 = 2l^2$, and so n^2 is even.

Following the same argument as for m^2 above, it follows that n is even.

Since m is even and n is even they have a common factor of 2. This contradicts the assumption that $\frac{m}{n}$ was in lowest terms.

Therefore $\sqrt{2}$ must be irrational.

Theorem 1.2 There are infinitely many primes. (/ in Euclid's *Elements*, Book IX, Proposition 20.)

For contradiction suppose there are only a finite number n of primes,

$$p_1, p_2, p_3, \ldots, p_n$$

Let $q = p_1 p_2 p_3 \cdots p_n + 1$.

Claim q is another prime distinct from the $p_1, p_2, \ldots, p_n$.

Clearly $q > p_i$ for every $i = 1, \ldots, n$ so it's distinct.

Is it prime? Why?