Theorem: The countable union of countable sets is countable

If I is countable and for every $\alpha \in I$ the set A_α is also countable, then

$$\bigcup_{\alpha \in I} A_\alpha \quad \text{is countable.}$$

Before this one more proposition

Proposition If $B \subseteq A$ and A is countable then B is countable.

Proposition: If A is countable and $f: B \to A$ is onto then B is countable.

If B is finite then it's clearly countable. Thus, assume that B is infinite.

axiom of choice allows you to find a_b.

For each $b \in B$ choose $a_b \in A$ such that $a_b \in f^{-1}(\{b\})$

or in otherwords so $f(a_b) = b$.

Define $g: B \to A$ as $g(b) = a_b$

Claim g is one-to-one. Suppose $g(x) = g(y)$ then $a_x = a_y$. Since $f(a_x) = x$ and $f(a_y) = y$ it follows that $x = y$

Therefore $g: B \to g(B)$ is a bijection. Thus $B \sim g(B)$

Since $g(B) \subseteq A$ and A is countable then $g(B)$ is countable. That means, B is countable. $\square$

Theorem: The countable union of countable sets is countable.
If I is countable and for every $\alpha \in I$ the set A_α is also countable, then

$$\bigcup_{\alpha \in I} A_\alpha \quad \text{is countable.}$$

Assume $I \neq \emptyset$ and $A_\alpha \neq \emptyset$ for each $\alpha \in I$

Since I is countable there is ~~an~~ a function $f : \mathbb{N} \to I$ onto

A_α is countable there is ~~an~~ a function $g_\alpha : \mathbb{N} \to A_\alpha$ onto

Since $\mathbb{N} \times \mathbb{N}$ is countable define $h : \mathbb{N} \times \mathbb{N} \to \bigcup_{\alpha \in I} A_\alpha$

as

$h(n,m) = g_{f(n)}(m)$ claim h is onto $\bigcup_{\alpha \in I} A_\alpha$

Let $x \in \bigcup_{\alpha \in I} A_\alpha$. Then by definition of union there is $\alpha \in I$ such that $x \in A_\alpha$

Since $f : \mathbb{N} \to I$ is onto there is $n \in \mathbb{N}$ such that $f(n) = \alpha$

since $g_\alpha : \mathbb{N} \to A_\alpha$ is onto there is $m \in \mathbb{N}$ such that $g_\alpha(m) = x$

Thus $h(n,m) = g_{f(n)}(m) = g_\alpha(m) = x$ ☑