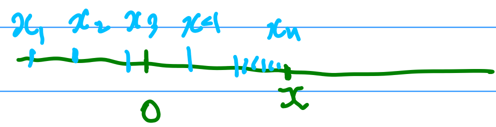
Define. A sequence $(x_n)_{\in\mathbb{N}}$ is said to be <u>bounded</u> if $\exists B > 0$ s.t. $|x_n| \leq B$ for all $n \in \mathbb{N}$.

Proposition: If $\lim\limits_{n\to\infty} x_n = x$ then $(x_n)_{n\in\mathbb{N}}$ is bounded.

Lemma: if $\lim\limits_{n\to\infty} x_n = x$ and $x \neq 0$ then there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that $|x_n| \geq \varepsilon$ for all $n \geq n_0$

Proof of Lemma:



(by def of limit)

Let $\varepsilon = \dfrac{|x|}{2}$. Since $\varepsilon > 0$ and $\lim\limits_{n\to\infty} x_n = x$ then there exists $n_0 \in \mathbb{N}$ s.t. $n \geq n_0$ implies $|x - x_n| < \varepsilon$.

mult by -1

Now estimate. Since $-|x - x_n| > -\varepsilon$

$$|x_n| = |x_n - x + x| \geq |x| - |x_n - x| > |x| - \varepsilon = |x| - \frac{|x|}{2} = \frac{|x|}{2} = \varepsilon \quad \square$$

Try to understand

What if $\varepsilon = \dfrac{|x|}{3}$ instead?

Let $\varepsilon = \dfrac{|x|}{3}$. Since $\varepsilon > 0$ and $\lim\limits_{n\to\infty} x_n = x$ then there exists $n_0 \in \mathbb{N}$ s.t. $n \geq n_0$ implies $|x - x_n| < \varepsilon$.

mult by -1

Now estimate. Since $-|x - x_n| > -\varepsilon$

$$|x_n| = |x_n - x + x| \geq |x| - |x_n - x| > |x| - \varepsilon = |x| - \frac{|x|}{3}$$

$$= 2\frac{|x|}{3} = 2\varepsilon > \varepsilon.$$

still worked $\square$

One more time what about $\varepsilon = \frac{2|x|}{3}$

Let $\varepsilon_0 = \frac{2|x|}{3}$. Since $\varepsilon_0 > 0$ and $\lim_{n\to\infty} x_n = x$ then and $\varepsilon = \varepsilon_0/2$.

there exists $n_0 \in \mathbb{N}$ s.t. $n \geq n_0$ implies $|x - x_n| < \varepsilon_0$.

mult by −1

Now estimate. Since $-|x-x_n| > -\varepsilon_0$

$$|x_n| = |x_n - x + x| \geq |x| - |x_n - x| > |x| - \varepsilon_0 = |x| - \frac{2|x|}{3}$$

$$= \frac{|x|}{3} = \frac{\varepsilon_0}{2} = \varepsilon.$$

still worked ☑

Proposition: If $\lim_{n\to\infty} x_n = x$ then $(x_n)_{n\in\mathbb{N}}$ is bounded.

Proof Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and let x be in $\mathbb{R}$ with $x_n \to x$. By Proposition 3.1 with $\varepsilon = 1$, there exists an n_0 in $\mathbb{N}$ such that $|x_n - x| < 1$ for all $n \geq n_0$. By Corollary 2.1, $|x_n| - |x| \leq |x_n - x| < 1$, which implies that $|x_n| < 1 + |x|$ for all $n \geq n_0$. Let $B = \max\{|x_1|, |x_2|, \ldots, |x_{n_0-1}|, 1 + |x|\}$. Then $B > 0$ and $|x_n| \leq B$ for all n in $\mathbb{N}$. ■

Let $\varepsilon = \frac{|x|}{2}$. Since $\varepsilon > 0$ and $\lim_{n\to\infty} x_n = x$ then

there exists $n_0 \in \mathbb{N}$ s.t. $n \geq n_0$ implies $|x - x_n| < \varepsilon$.

$$|x_n| = |x_n - x + x| \leq |x_n - x| + |x| < \varepsilon + |x| = \frac{|x|}{2} + |x| = \frac{3|x|}{2}$$

Take $B = \max\{|x_1|, |x_2|, \ldots, |x_{n_0-1}|, \frac{3|x|}{2}\}$. Then $B > 0$

and $|x_n| \leq B$ for all $n \in \mathbb{N}$.

Limit Laws

Theorem 3.2 Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}$; let x and y be in $\mathbb{R}$ with $x_n \to x$ and $y_n \to y$. Then

1. $\lim_{n\to\infty} (x_n + y_n) = x + y = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$;
2. $\lim_{n\to\infty} x_n y_n = xy = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} y_n$;
3. $\lim_{n\to\infty} cx_n = cx = c \lim_{n\to\infty} x_n$ for all real numbers c;
4. $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$, provided $y_n \neq 0$ for all n in $\mathbb{N}$ and $y \neq 0$.

Proof of 1

Let $\varepsilon > 0$

Suppose $x_n \to x$ and $y_n \to y$. Let $\delta \in (0,1)$

Let $\varepsilon_1 = \delta\varepsilon > 0$. Then by definition of limit there is $n_1 \in \mathbb{N}$ such that $n \geq n_1$ implies $|x - x_n| < \varepsilon_1$

Let $\varepsilon_2 = (1-\delta)\varepsilon > 0$. Then by definition of limit there is $n_2 \in \mathbb{N}$ such that $n \geq n_2$ implies $|y - y_n| < \varepsilon_2$.

Claim 1. $\lim_{n\to\infty} (x_n + y_n) = x + y$

need to show there is n_0 such that

$|x_n + y_n - (x+y)| < \varepsilon$ for all $n \geq n_0$

Estimate, Let $n_0 = \max(n_1, n_2)$ then for $n \geq n_0$ holds

$$|x_n + y_n - (x+y)| \leq |x_n - x| + |y_n - y| < \varepsilon_1 + \varepsilon_2 \leq \varepsilon$$

Claim 2. $\lim\limits_{n\to\infty} x_n y_n = xy = \lim\limits_{n\to\infty} x_n \cdot \lim\limits_{n\to\infty} y_n$;

Let $\varepsilon > 0$

Suppose $x_n \to x$ and $y_n \to y$.

Let $\varepsilon_1 = \boxed{\qquad} > 0$. Then by definition of limit there is $n_1 \in \mathbb{N}$ such that $n \geq n_1$ implies $|x - x_n| < \varepsilon_1$

Let $\varepsilon_2 = \boxed{\qquad} > 0$. Then by definition of limit there is $n_2 \in \mathbb{N}$ such that $n \geq n_2$ implies $|y - y_n| < \varepsilon_2$.

Need to show there is n_0 such that $|x_n y_n - xy| < \varepsilon$ for all $n \geq n_0$

Let $n_0 = \max\{n_1, n_2\}$

Estimate

$$|x_n y_n - xy| \leq |x_n y_n \overbrace{- x_n y + x_n y}^{\text{intermediate point of comparison.}} - xy|$$