Theorem 3.11 (Bolzano-Weierstrass Theorem for sets) Every bounded infinite subset of $\mathbb{R}$ has an accumulation point in $\mathbb{R}$.

But First

Proposition: If x is an accumulation point of A then every neighborhood of x contains infinitely many points of A.

recall

What's an accumulation point?

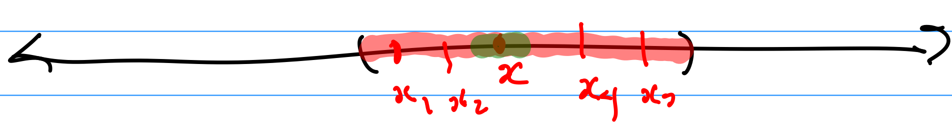
Let $A \subseteq \mathbb{R}$. Then $x \in \mathbb{R}$ is an accumulation point of A is every neighborhood of x contains at least one point in A other than x.

Proof:

For contradiction suppose that there was a neighborhood U of x such that $U \cap A$ had only finite number of points.

Thus,

$$(U \cap A) \setminus \{x\} = \{x_1, x_2, \dots, x_n\} \quad \text{where} \quad x_i \in A$$
$$\text{and} \quad x_i \neq x \text{ for all } i.$$



Let $\varepsilon = \min\{|x_1 - x|, |x_2 - x|, \dots, |x_n - x|\}$

Since $|x_i - x| > 0$ for all i and there are a finite number of terms in the set, then the minimum exists. So there is $i_0 \in \mathbb{N}$ so that

$$\varepsilon = |x_{i_0} - x|.$$

Consider the neighborhood $(x-\frac{\varepsilon}{2}, x+\frac{\varepsilon}{2})$. Then
Then this neighborhood has no points in A, except possibly x itself.

Note: Why does $(x-\frac{\varepsilon}{2}, x+\frac{\varepsilon}{2})$ not contain any other points in A:

If there was a $y \in A \cap (x-\frac{\varepsilon}{2}, x+\frac{\varepsilon}{2})$ with $y \neq x$ then $y = x_k$ for some k by definition of the x_i's.

Then $|y-x| < \frac{\varepsilon}{2}$ but also

$$|y-x| = |x_k - x| \geq \min\{|x_i - x| : i=1,\dots,n\} = \varepsilon$$

Since $\varepsilon < \frac{\varepsilon}{2}$ is not true then $(x-\frac{\varepsilon}{2}, x+\frac{\varepsilon}{2})$ does not contain any point of A other than possibly x.

This contradicts x being an accumulation point. $\boxtimes$

Theorem 3.11 (Bolzano-Weierstrass Theorem for sets) Every bounded infinite subset of $\mathbb{R}$ has an accumulation point in $\mathbb{R}$.

Let $A \subseteq \mathbb{R}$

bounded

infinite

Since A is infinite it contains a sequence of distinct points $(x_n)_{n\in\mathbb{N}}$.

Since A is bounded thus $(x_n)_{n\in\mathbb{N}}$ is bounded.
By the Bolzano-Weierstrass theorem there exist a convergence subsequence $(x_{n_k})_{k\in\mathbb{N}}$.

Thus $x_{n_k} \to x$ for some $x \in \mathbb{R}$.

Claim x is an accumulation point of A.

Let U be a neighborhood of x. Need to show that $U \cap A$ contains at least one other point than x.
Or by the proposition that $U \cap A$ contains infinitely many points.

Since $x_{n_k} \to x$ there is n_0 such that $k \geq n_0$ implies $x_{n_k} \in U$. (definition of convergence.)

That is the sequence $(x_{n_k})_{k\in\mathbb{N}}$ is eventually in any neighborhood of its limit point.

Since $x_{n_k} \in A$ for all k. Then

$$x_{n_k} \in A \cap U \text{ for all } k \geq n_0$$

and since the x_{n_k} are distinct that's an infinite number of points $\square$

Q ; Let $\lim_{n\to\infty} x_n = x$ and $A = \{x_1, x_2, \dots\}$

Then is x an accumulation point of A?

No, Suppose $x_n = 3$ for all n. Then $\lim_{n\to\infty} x_n = 3$

but $A = \{3\}$ and then 3 is an isolated point of A

Cauchy Sequences.

Definition: $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ is a Cauchy sequence if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $m, n \geq n_0$ implies $|x_n - x_m| < \varepsilon$.

① Bolzano Weierstrass and Monotone convergence are ways to find the limit.

② Cauchy sequences are a way to talk about convergence without finding the limit.

Proposition 3.5 A convergent sequence is Cauchy.

Lemma 3.3 A Cauchy sequence is bounded.

Theorem 3.12 A sequence in $\mathbb{R}$ is Cauchy if and only if the sequence converges.