Recall $\mathbb{R}^{\#} = \mathbb{R} \cup \{\infty, -\infty\}$

Talking about convergence using technical terms such as
neighborhood and eventually.

Let $x \in \mathbb{R}$ then a neighborhood of x is
$(x-\varepsilon, x+\varepsilon)$ for $\varepsilon > 0$

A sequence $(x_n)_{n \in \mathbb{N}}$ is eventually in a neighborhood U if there is n_0 such that $x_n \in U$ for $n \geq n_0$.

Thus $x_n \to x$ means for every neighborhood U of x that $(x_n)_{n \in \mathbb{N}}$ is eventually in U.

To understand limits at infinity all we need to do is define neighborhoods at infinity.

Definition 3.12 Let α and β be in $\mathbb{R}$. The open ray

$$(\alpha, \infty) = \{x \in \mathbb{R} : x > \alpha\}$$

is a *neighborhood of* ∞, while the open ray

$$(-\infty, \beta) = \{x \in \mathbb{R} : x < \beta\}$$

is a *neighborhood of* $-\infty$.

Thus $x_n \to \infty$ means for every neighborhood U of ∞ that $(x_n)_{n \in \mathbb{N}}$ is eventually in U.

How many theorems still hold for $\mathbb{R}^{\#}$.

Theorem 3.13 Limits of sequences are unique.

Theorem 3.14 Let $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$, and $(z_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}$. Let x be in $\mathbb{R}^{\#}$ and suppose that $x_n \to x$, $y_n \to \infty$, and $z_n \to -\infty$.

1. If $-\infty < x \leq \infty$, then $x_n + y_n \to \infty$.
2. If $-\infty \leq x < \infty$, then $x_n + z_n \to -\infty$.
3. If $0 < x \leq \infty$, then $x_n y_n \to \infty$ and $x_n z_n \to -\infty$.
4. If $-\infty \leq x < 0$, then $x_n y_n \to -\infty$ and $x_n z_n \to \infty$.
5. If x is in $\mathbb{R}$, then $\dfrac{x_n}{y_n} \to 0$ and $\dfrac{x_n}{z_n} \to 0$.

Theorem 3.15 Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$; let x be in $\mathbb{R}^{\#}$ with $x_n \to x$. Then every subsequence of $(x_n)_{n\in\mathbb{N}}$ has limit x.

Proposition 3.6 Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Then

1. $\lim_{n\to\infty} x_n \neq \infty$ if and only if $(x_n)_{n\in\mathbb{N}}$ has a subsequence that is bounded above;
2. $\lim_{n\to\infty} x_n \neq -\infty$ if and only if $(x_n)_{n\in\mathbb{N}}$ has a subsequence that is bounded below.

This does not imply the limit exists... it means not $\left(\lim_{n\to\infty} x_n = \infty\right)$

$\lim_{n\to\infty} x_n = \infty$ means for every neighborhood U of ∞ that $(x_n)_{n\in\mathbb{N}}$ is eventually in U.

means for every $a \in \mathbb{R}$ than there is $n_0 \in \mathbb{N}$ such that $x_n \geq a$ for all $n \geq n_0$.

not $\left(\lim_{n\to\infty} x_n = \infty\right)$ means not (for every $a \in \mathbb{R}$ than there is $n_0 \in \mathbb{N}$ such that $x_n \geq a$ for all $n \geq n_0$.)

means there exist $a \in \mathbb{R}$ such that for all $n_0 \in \mathbb{N}$ such that there exists $n \geq n_0$ for which $x_n < a$.

Theorem 3.16 If $(x_n)_{n\in\mathbb{N}}$ is a monotone sequence in $\mathbb{R}$, then $(x_n)_{n\in\mathbb{N}}$ has a limit in $\mathbb{R}^{\#}$.

8. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Show that

☑ (a) if $(x_n)_{n\in\mathbb{N}}$ is unbounded above, then $(x_n)_{n\in\mathbb{N}}$ has a subsequence with limit ∞;

(b) if $(x_n)_{n\in\mathbb{N}}$ is unbounded below, then $(x_n)_{n\in\mathbb{N}}$ has a subsequence with limit $-\infty$. [*Hint:* See Exercise 5 in Section 3.3.]

Let $(x_n)_{n\in\mathbb{N}}$ be unbounded above. Take $n_1 = 1$

Then since the sequence is unbounded there is $n_2 > n_1$ such that $x_{n_2} > x_{n_1} + 1$.

Then since the sequence is unbounded there is $n_{k+1} > n_k$ such that $x_{n_{k+1}} > x_{n_k} + 1$.

Note $x_{n_k} \geq x_{n_1} + (k-1) \to \infty$ as $k \to \infty$. ✓

Given a sequence $(x_n)_{n\in\mathbb{R}}$

$$E = \{ x \in \mathbb{R}^{\#} : x_{n_k} \to x \text{ for some subsequence of } (x_n)_{n\in\mathbb{N}} \}$$

$$\liminf_{n\to\infty} x_n = \inf E$$

$$\limsup_{n\to\infty} x_n = \sup E.$$

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty}\left(\inf_{m\geq n} x_m\right)$$

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty}\left(\sup_{m\geq n} x_m\right)$$

alternative definitions from Wikipedia..

These definitions are the same... which is easier to visualize?

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty}\left(\inf_{m\geq n} x_m\right)$$

$$\inf_{m\geq n} x_m = \inf \underbrace{\{x_m : m \geq n\}}$$

as n gets bigger this set gets smaller
and the inf gets larger...

Note if. $A \subseteq B$ then $\inf A$? $\inf B$

① $\leq$

② $\geq$

$\{1\} \subseteq \{1, -15\}$

Therefore $\inf_{m \geq n} x_m$ is monotone increasing

so $\lim_{n \to \infty} \inf_{m \geq n} x_m$ exists by Theorem 3.16

Also $\sup_{m \geq n} x_n$ is if increasing or decreasing.

Note if. $A \subseteq B$ then $\sup A$? $\sup B$

$\leq$

Proposition 3.7 Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}$ and let x be in $\mathbb{R}^{\#}$. Then $x_n \to x$ if and only if $\limsup x_n = \liminf x_n = x$.

Let $E = \{x : x_{n_k} \to x$ for some subsequence$\}$

if $\sup E = \inf E = x$ then $E = \{x\}$.

Read the proof