Exercise 3.4 #9

9. Let A be a nonempty bounded subset of $\mathbb{R}$ with $\alpha = \sup A$ and $\beta = \inf A$. Show that A contains a monotone increasing sequence with limit α and that A contains a monotone decreasing sequence with limit β. [*Hint:* By Theorem 3.9 it suffices to find a sequence in A with limit α. Consider two cases: α in A and α in $\mathbb{R} \setminus A$.]

recall

Theorem 3.9 (Monotone Subsequence Theorem) Every sequence in $\mathbb{R}$ has a monotone subsequence.

this was the proof with peaks

Proof Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. For the purpose of this proof, we call the mth term x_m a *peak* if $x_m \geq x_n$ for all $n \geq m$. That is, x_m is a peak if x_m is never exceeded by any term that follows it.

9. Let A be a nonempty bounded subset of $\mathbb{R}$ with $\alpha = \sup A$ and $\beta = \inf A$. Show that A contains a monotone increasing sequence with limit α and that A contains a monotone decreasing sequence with limit β. [*Hint:* By Theorem 3.9 it suffices to find a sequence in A with limit α. Consider two cases: α in A and α in $\mathbb{R} \setminus A$.]

?

Proof

Case $\alpha \in A$: Then take $x_n = \alpha$ for all n. Note that the constant sequence is monotone increasing (or decreasing) and $x_n \to \alpha$.

Case $\alpha \notin A$. Let $\varepsilon_n = \frac{1}{n}$. Note $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ as $n \to \infty$, since α is the least upper bound of A, then $\alpha - \varepsilon_n$ is not an upper bound, Thus there is $x_n \in A$ such that

$\alpha - \varepsilon_n < x_n$ since $\alpha \notin A$ then also $x_n < \alpha$.

Claim $x_n \to \alpha$ as $n \to \infty$.

Why? Suppose $\varepsilon > 0$. Then by the Archimedian principle there is $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \varepsilon$.

Note $\alpha - \varepsilon_n < x_n < \alpha$ implies $\boxed{\varepsilon_n - \alpha > -x_n} > -\alpha$

Then for $n \geq n_0$ we have

$$|x_n - \alpha| = \alpha - x_n < \alpha - (\varepsilon_n - \alpha) = \varepsilon_n = \frac{1}{n} \leq \frac{1}{n_0} < \varepsilon.$$

By Theorem 3.9 $(x_n)_{n\in\mathbb{N}}$ has a monotone subsequence $(x_{n_k})_{k\in\mathbb{N}}$. Claim $(x_{n_k})_{k\in\mathbb{N}}$ is increasing.

Suppose, for contradiction, it were decreasing. Note since $(x_{n_k})_{k\in\mathbb{N}}$ is monotone it must be either monotone increasing, monotone decreasing (or both).

Then $x_{n_k} \leq x_{n_1}$ for all $k \in \mathbb{N}$

Since $x_{n_1} < \alpha$ then $-\alpha < -x_{n_1} \leq -x_{n_k}$

$$|x_{n_k} - \alpha| = \alpha - x_{n_k} \geq \underbrace{\alpha - x_{n_1}}_{\varepsilon = \alpha - x_{n_1}} > 0$$

Which means $\lim_{k\to\infty} x_{n_k} \neq \alpha$. But I know it converges

So the only alternative is that $(x_{n_k})_{k\in\mathbb{N}}$ is increasing.

Proposition 3.9 Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Then $\limsup x_n$ and $\liminf x_n$ are both in E.

read for next time...

Change notation so it makes sense to you...