Proposition 3.9 Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Then $\limsup x_n$ and $\liminf x_n$ are both in E.

read for next time

$$E = \{x \in \mathbb{R}^{\#} : \text{there exist a subsequence such that } x_{n_k} \to x\}$$

$$\lim_{n\to\infty} \sup_{k\geq n} x_k = \limsup_{n\to\infty} x_n = \sup E = \max E$$

$$\lim_{n\to\infty} \inf_{k\geq n} x_k = \liminf_{n\to\infty} x_n = \inf E = \min E$$

Proof: Let $\alpha = \sup E$.

Case $\alpha \in \mathbb{R}$.

recall

9. Let A be a nonempty bounded subset of $\mathbb{R}$ with $\alpha = \sup A$ and $\beta = \inf A$. Show that A contains a monotone increasing sequence with limit α and that A contains a monotone decreasing sequence with limit β. [*Hint:* By Theorem 3.9 it suffices to find a sequence in A with limit α. Consider two cases: α in A and α in $\mathbb{R} \setminus A$.]

By Exercise 3.4#9 there exist a sequence $(y_j)_{j\in\mathbb{N}}$ in E that is monotone increasing and $y_j \to \alpha$ as $j \to \infty$.

since $\alpha < \infty$ then $y_j < \infty$ for all j.

Since $y_j \in E$ then there is a subsequence such that

recall

$$E = \{x \in \mathbb{R}^{\#} : \text{there exist a subsequence such that } x_{n_k} \to x\}$$

$x_{n_{j,k}} \to y_j$ as $k \to \infty$

By definition of convergence since $y_j \in \mathbb{R}$ then for each $\varepsilon_j = \frac{1}{j} > 0$

there exist k_j such that $k \geq k_j$ implies $|x_{n_{j,k}} - y_j| < \varepsilon_j$.

There is a diagram in the book describing whats going on

$x_{n,1}$

$$
\begin{array}{ccccccc}
x_{1,n_1} & x_{1,n_2} & x_{1,n_3} & \cdots & \to & y_1 \\
x_{2,n_1} & x_{2,n_2} & x_{2,n_3} & \cdots & \to & y_2 \\
\vdots & \vdots & & & & \vdots \\
x_{i,n_1} & x_{i,n_2} & x_{i,n_3} & \cdots & \to & y_i \\
\vdots & \vdots & & & & \vdots \\
 & & & & & \downarrow \\
 & & & & & \alpha
\end{array}
$$

Want to construct a subsequence x_{n_j} that converges to α. by moving along the diagonal.

Let $n_1 = n_{1,k_1}$ then $|x_{n_1} - y_1| < \varepsilon_1 = \frac{1}{1}$

$n_2 = n_{2,k}$ where $k \geq k_2$ and $n_{2,k} \geq n_1 + 1$

$\vdots$ then $|x_{n_2} - y_2| < \varepsilon_2 = \frac{1}{2}$ and $n_1 < n_2$

$n_{j+1} = n_{j+1,k}$ where $k \geq k_{j+1}$ and $n_{j+1,k} \geq n_j + 1$

then $|x_{n_{j+1}} - y_{j+1}| < \varepsilon_{j+1} = \frac{1}{j+1}$ and $n_j < n_{j+1}$

Claim $x_{n_j} \to \alpha$ as $j \to \infty$.

Let $\varepsilon > 0$. Since $y_j \to \alpha$ then choose j_1 so large that $j \geq j_1$ implies $|y_j - \alpha| < \varepsilon/2$. Also choose j_2 so large that $\frac{1}{j_2} < \varepsilon/2$ by archimedian principle. Let $j_0 = \max(j_1, j_2)$ then $j \geq j_0$ implies $j \geq j_1$ so $|y_j - \alpha| < \varepsilon/2$ and

$$|x_{n_j} - \alpha| \leq |x_{n_j} - y_j| + |y_j - \alpha| < \frac{1}{j} + \varepsilon/2 \leq \frac{1}{j_2} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore $x_{n_j} \to \alpha$ and we've done. This means $\alpha \in E$.

Case $\alpha = -\infty$

Then $\sup E = -\infty$ thus $E = \{-\infty\}$ in which case $\alpha \in E$.

Case $\alpha = \infty$ then what?

Read the proof in the book to see how this case works...
Use exercise 3.7#11 instead of 3.4#9.

Quiz on Friday after spring break over homework that will be posted soon...

Chapter 4: Continuity...

Definition: Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$ and $c \in D$.
Then f is continuous at c means

For every neighborhood V of $f(c)$ there is a neighborhood U of c such that $x \in D \cap U$ implies $f(x) \in V$.

alternatively

For every $\varepsilon > 0$ there is a $\delta > 0$ such that $x \in D$ and $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$.

Theorem: Composition of continuous functions is continuous.
Namely: Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ such that $f(A) \subseteq B$. Suppose f is continuous at $c \in A$ and g is continuous at $f(c)$ then $g \circ f: A \to \mathbb{R}$ is continuous at c.