Theorem 4.1 Let $f : D \to \mathbb{R}$ and let c be in D. Then f is continuous at c if and only if whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in D that converges to c, then $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(c)$.

Suppose $f: D \to \mathbb{R}$ and $c \in D$.

"$\Rightarrow$" Let f be continuous at c. Want to show that $f(x_n) \to f(c)$ whenever $x_n \in D$ and $x_n \to c$.

For contradiction, suppose not. Thus, there is ... $f(x_n) \to f(c)$

Theorem 4.1 Let $f : D \to \mathbb{R}$ and let c be in D. Then f is continuous at c if and only if whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in D that converges to c, then $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(c)$.

not needed

$\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}$ s.t. $\forall n \geq n_0$ implies $|f(x_n) - f(c)| < \varepsilon$

Thus $f(x_n) \not\to f(c)$ means

$\exists \varepsilon > 0\ \forall n_0 \in \mathbb{N}\ \exists n \geq n_0$ s.t. $|f(x_n) - f(c)| \geq \varepsilon$

(*) Since f is continuous at c there is $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$.

Since $\delta > 0$ and $x_n \to c$ there is n_1 such that $n \geq n_1$ implies $|x_n - c| < \delta$.

Since $|x_n - c| < \delta$ then $|f(x_n) - f(c)| < \varepsilon$ by (*) which implies f is continuous..

"$\Leftarrow$" Suppose that for all sequences $(x_n)_{n\in\mathbb{N}}$ in D such that $x_n \to c$ we have that $f(x_n) \to f(c)$ as $n \to \infty$. Claim that f is continuous at c.

For contradiction suppose f were not continuous at c.

Continuity at c means

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t $x \in D$ and $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$

not Continuity at c means

(**) $\exists \varepsilon > 0 \ \forall \delta > 0 \ \exists x \in D$ and $|x - c| < \delta$ s.t. $|f(x) - f(c)| \geq \varepsilon$

Chose $\varepsilon > 0$ as in (**).

for $\delta = \frac{1}{n}$ there is $x_n \in D$ with $|x_n - c| < \frac{1}{n}$ s.t $\boxed{|f(x_n) - f(c)| \geq \varepsilon}$.

By hypothesis

Suppose that for all sequences $(x_n)_{n \in \mathbb{N}}$ in D such that $x_n \to c$ we have that $f(x_n) \to f(c)$ as $n \to \infty$.

Since $x_n \to c$ and $x_n \in D$ then by hypothesis $f(x_n) \to f(c)$ but that contradicts $\boxed{|f(x_n) - f(c)| \geq \varepsilon}$.

Therefore f is continuous at c

Proposition 4.4 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ with $f(x) = 0$ for all x in $\mathbb{Q}$. Then $f(x) = 0$ for all x in $\mathbb{R}$.

proof. Let $c \in \mathbb{R} \setminus \mathbb{Q}$. Let $x_n \in \mathbb{Q}$ such that $x_n \to c$. Since f is continuous then $f(x_n) \to f(c)$. But $f(x_n) = 0$ since $x_n \in \mathbb{Q}$. Therefore $0 \to f(c)$ implies $f(c) = 0$.

<u>Remark</u>; to see there is $x_n \in \mathbb{Q}$ with $x_n \to c$ not that $\mathbb{Q}$ is dense in $\mathbb{R}$ so for each neighborhood $(c-\frac{1}{n}, c+\frac{1}{n})$ we have $(c-\frac{1}{n}, c+\frac{1}{n}) \cap \mathbb{Q} \neq \emptyset$ so choose $x_n \in (c-\frac{1}{n}, c+\frac{1}{n}) \cap \mathbb{Q}$

Proposition 4.5 Let $f : D \to \mathbb{R}$ with c in D. If f is continuous at c, then there is a neighborhood U of c such that f is bounded on $U \cap D$.

For contradiction, suppose for every neighborhood U of c that f is unbounded on $U \cap D$.

For $n \in \mathbb{N}$ let $U = (c-\frac{1}{n}, c+\frac{1}{n})$, thus f is unbounded on $(c-\frac{1}{n}, c+\frac{1}{n}) \cap D$. So there is $x_n \in (c-\frac{1}{n}, c+\frac{1}{n}) \cap D$ such that $|f(x_n)| \geq n$

as big as I like...

Since f is continuous and $x_n \in D$ with $x_n \to c$, then we have $f(x_n) \to f(c)$. But $|f(x_n)| \geq n$ implies that's impossible. Contradiction!

Therefore

there is a neighborhood U of c such that f is bounded on $U \cap D$.

Lemma 4.2 Let $f : D \to \mathbb{R}$ with c in D. Make the following suppositions:

1. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{Q} \cap D$ converging to c, then $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(c)$;

and

2. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in $(\mathbb{R} \setminus \mathbb{Q}) \cap D$ converging to c, then $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(c)$.

Then f is continuous at c.

To show f is continuous at c. For all $x_n \in D$ and $x_n \to c$ we need to show $f(x_n) \to f(c)$.

For contradiction suppose there is a sequence $x_n \in D$ with $x_n \to c$ such that $\boxed{\text{not } f(x_n) \to f(c)}$.

Thus $\exists\, \varepsilon > 0$ and a sequence such that $\boxed{|f(x_n) - f(c)| \geq \varepsilon}$.

?

next time finish this proof.

Define

$$A = \{ n \in \mathbb{N} : x_n \in \mathbb{Q} \} \qquad B = \{ n \in \mathbb{N} : x_n \in \mathbb{R} \setminus \mathbb{Q} \}$$

Since $A \cup B = \mathbb{N}$ then either A or B or both are infinite.

Case A is infinite. Then there is a subsequence x_{n_k} such that $x_{n_k} \in \mathbb{Q}$ and $x_{n_k} \to c$ as $k \to \infty$.

By hypothesis

1. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{Q} \cap D$ converging to c, then $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(c)$;

we have $f(x_{n_k}) \to f(c)$.