Theorem 3.3 Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in the closed interval $[a, b]$. Let x be in $\mathbb{R}$ with $x_n \to x$. Then x is in $[a, b]$.

Proposition 3.2 A convergent sequence is bounded.

Theorem 3.10 (Bolzano-Weierstrass Theorem for sequences) A bounded sequence in $\mathbb{R}$ has a convergent subsequence (that is, a subsequence that converges to a real number).

(*) **Theorem 3.13** Limits of sequences are unique.

We now recall Definition 3.10, Proposition 3.4, and Exercise 6 in Section 3.5. For $D \subset \mathbb{R}$ and c in $\mathbb{R}$,

c is an accumulation point of D

if and only if every neighborhood of c contains a point of D different from c

if and only if every neighborhood of c contains infinitely many points of D

if and only if there exists a sequence of distinct points in D converging to c.

Define The limit of a function

note $c = \infty$ might be allowed here.

Let $f: D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$ and c an accumulation point of D. Then the limit of f at c is $L \in \mathbb{R}^{\#}$ means for every neighborhood V of L there is a neighborhood U of c such that $x \in U \cap D$ and $x \neq c$ implies $f(x) \in V$.

Notation $\lim_{x \to c} f(x) = L$ or $f(x) \to L$ as $x \to c$.

$f: D \to \mathbb{R}$, $\subseteq \mathbb{R}$, $\in \mathbb{R}$, $\in \mathbb{R}^{\#}$

Proposition 4.6 Let D, f, c, and L be as in Definition 4.3. Then $\lim_{x\to c} f(x) = L$ if and only if $\lim_{n\to\infty} f(x_n) = L$ for all sequences $(x_n)_{n\in\mathbb{N}}$ in D such that $x_n \neq c$ for all n and $\lim_{n\to\infty} x_n = c$.

"$\Rightarrow$" Suppose $\lim_{x\to c} f(x) = L$. Claim $\lim_{n\to\infty} f(x_n) = L$ for all sequences in D such that $x_n \neq c$ and $x_n \to c$ as $n \to \infty$

~~Let $\varepsilon > 0$~~ Let V be a neighborhood of L.

Since $\lim_{x\to c} f(x) = L$ there is a neighborhood U of c such that $x \in U \cap D$ and $x \neq c$ implies $f(x) \in V$.

Let $x_n \in D$ such that $x_n \neq c$ and $x_n \to c$ as $n \to \infty$.

Since $x_n \to c$ and U is a neighborhood of c, then $(x_n)_{n\in\mathbb{N}}$ is eventually in U. Thus, there is $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $x_n \in U$.

Since $x_n \in D$ then $x_n \in U \cap D$ for all $n \geq n_0$

By hypothesis $x_n \neq c$. Therefore $f(x_n) \in V$ for all $n \geq n_0$

In particular $f(x_n)$ is eventually in V.

Therefore $(f(x_n))_{n\in\mathbb{N}}$ converges to c.

"$\Leftarrow$" Suppose $\lim_{n\to\infty} f(x_n) = L$ for all sequences in D such that $x_n \neq c$ and $x_n \to c$ as $n \to \infty$ Claim $\lim_{x\to c} f(x) = L$

For contradiction suppose not $\lim_{x \to c} f(x) = L$

note
$\lim_{x \to c} f(x) \neq L$
probably means the same thing.

for every neighborhood V of L there is a neighborhood U of c such that $x \in U \cap D$ and $x \neq c$ implies $f(x) \in V$.

There exists a neighborhood V of L such that for all neighborhoods U of c there exists $x \in U \cap D$ with $x \neq c$ such that $f(x) \notin V$.

Let $U = (c - \frac{1}{n}, c + \frac{1}{n})$ and choose $x_n \in (c - \frac{1}{n}, c + \frac{1}{n}) \cap D$ with $x_n \neq c$ such that $f(x_n) \notin V$. Note, since c is an accumulation point there is always at least one such x_n.

Then $x_n \to c$ as $n \to \infty$ and $x_n \neq c$ so by hypothesis $f(x_n) \to L$ as $n \to \infty$.
Thus $(f(x_n))_{n \in \mathbb{N}}$ is eventually in any neighborhood of L.
In particular there is $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $f(x_n) \in V$.
This is a contradiction. Therefore $\lim_{x \to c} f(x) = L$.

Corollary 4.1 If f has a limit at c, then this limit is unique.

because of (*).

Proposition 4.7 For $D \subset \mathbb{R}$, c an accumulation point of D, f and g two real valued functions on D, L_1 and L_2 in $\mathbb{R}^{\#}$, suppose that

$$\lim_{x \to c} f(x) = L_1 \text{ and } \lim_{x \to c} g(x) = L_2.$$

Then

1. $\lim_{x \to c} (f(x) + g(x)) = L_1 + L_2$;
2. $\lim_{x \to c} f(x)g(x) = L_1 L_2$; $|f(x)g(x) - L_1L_2| \leq |f(x)g(x) - f(x)L_2| + |f(x)L_2 - L_1L_2|$
3. $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L_1}{L_2}$, if $L_2 \neq 0$

provided the right members of parts 1, 2, and 3 are defined. Note that $\infty - \infty$, $0 \cdot \infty$, ∞/∞, and $L_1/0$ are not defined.

because limit laws also hold for sequences ...

Example 4.14 Let $f(x) = x \sin(1/x)$ for $x \neq 0$.

Claim $\lim_{x \to 0} f(x) = 0$.

For every neighborhood V of 0. ... means.

Let $\varepsilon > 0$. Then choose a neighborhood U.

Choose $\delta = \varepsilon$ then $|x - 0| < \delta$ implies

$$|f(x) - 0| = |x \sin(1/x)| \leq |x| < \delta = \varepsilon.$$