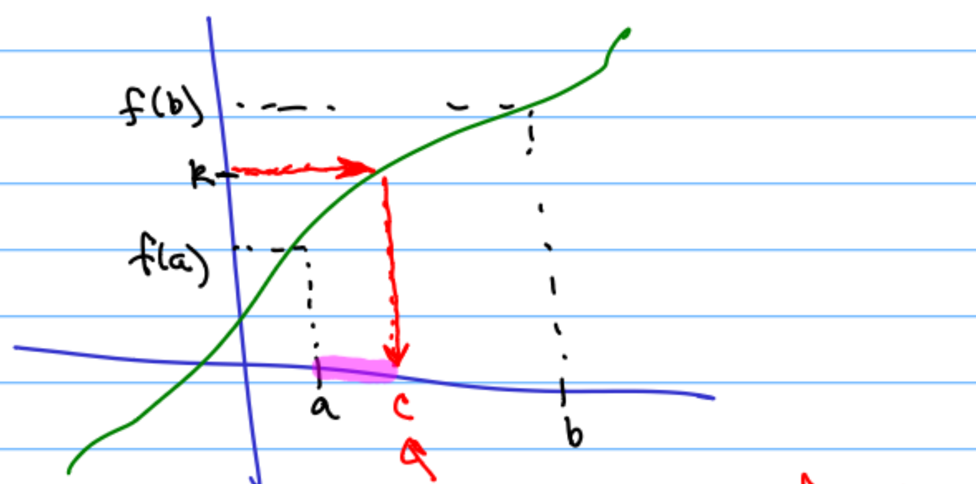
Exam on Monday:

- Everything in Chapters 1–3 and homework
- Definitions in Chapter 4 and proofs of two named theorems to be named on Friday...

Theorem 4.3 (Intermediate Value Theorem) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Assume that $f(a) \neq f(b)$. Then, for any k between $f(a)$ and $f(b)$, there is a c in $[a, b]$ such that $f(c) = k$.



For definiteness assume $f(a) < f(b)$. If not then consider the negatives of these functions in their place.

Let $S = \{x : f(x) < k\}$

since k is between $f(a)$ and $f(b)$ then $f(a) < k < f(b)$.

Thus $a \in S$. So $S \neq \emptyset$ and S is bounded. By the completeness axiom $c = \sup S \in \mathbb{R}$.

Claim $a < c < b$.

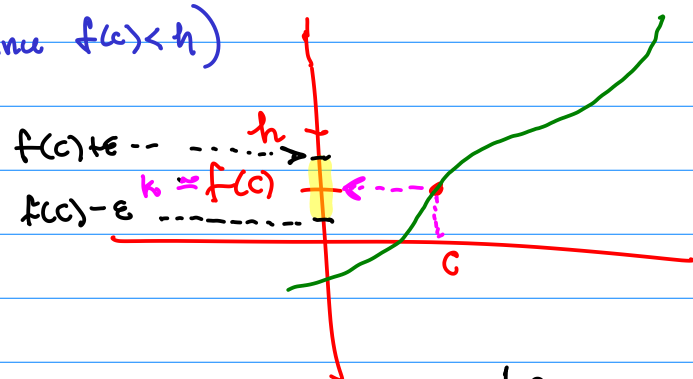
① $f(a) < k$ Thus $\exists\, \delta > 0$ s.t. $f(x) < k$ for all $x \in (a-\delta, a+\delta) \cap [a, b]$.

This is from HW §4.1#4

4. Let $f : D \to \mathbb{R}$ be continuous at c in D. Let h be in $\mathbb{R}$ with $f(c) < h$ [respectively, $f(c) > h$]. Show that there is a neighborhood U of c such that if x is in $U \cap D$, then $f(x) < h$ [respectively, $f(x) > h$].

Solution to HW.

Let $\varepsilon = \dfrac{h - f(c)}{2} > 0$ (since $f(c) < h$)



Since f is continuous at c there is $\delta > 0$ such that $|x - c| < \delta$ and $x \in D$ implies $|f(x) - f(c)| < \varepsilon$.

Thus, for $|x - c| < \delta$ and $x \in D$ we have

$$f(x) = f(x) - f(c) + f(c) \leq |f(x) - f(c)| + f(c)$$

$$< \varepsilon + f(c) = \frac{h - f(c)}{2} + f(c) = \frac{h}{2} + \frac{f(c)}{2} < \frac{h}{2} + \frac{h}{2} = h$$

Remark: if only $f(c) \leq h$ there may not be a neighborhood of c where this same inequality holds

Let $S = \{x : f(x) < k\}$ where $f(a) < k < f(b)$, $c = \sup S$.

Claim $a < c < b$.

① $f(a) < k$ Thus $\exists\, \delta > 0$ s.t. $f(x) < k$ for all $x \in (a - \delta, a + \delta) \cap [a, b]$.

Thus $f(a + \frac{\delta}{2}) < k$ so $a + \frac{\delta}{2} \in S$ so $a + \frac{\delta}{2} \leq c$ since c is an upper bound

Thus $a < c$.

② Since $k < f(b)$ $\exists \delta > 0$ s.t. $f(x) > k$ for all $x \in (b-\delta, b+\delta) \cap [a,b]$.
Thus $b-\delta$ is also an upper bound. Since $x \in (b-\delta, b+\delta) \cap [a,b]$ implies $x \notin S$.
Since c is least upper bound $c \leq b-\delta$ so $c < b$

Consequently $a < c < b$.

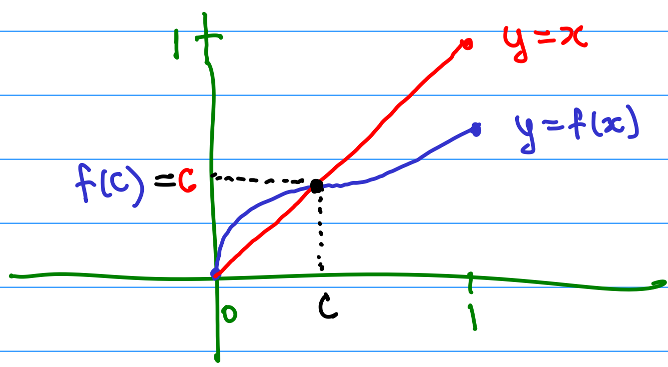
Claim that $f(c) = k$. For contradiction suppose not.

Case $f(c) < k$. $\exists \delta > 0$ s.t. $f(x) < k$ for all $x \in (c-\delta, c+\delta) \cap [a,b]$.
Thus $f(c+\frac{\delta}{2}) < k$. so $c+\frac{\delta}{2} \in S$ and this contradicts c being an upper bound.

Case $f(c) > k$, $\exists \delta > 0$ s.t. $f(x) > k$ for all $x \in (c-\delta, c+\delta) \cap [a,b]$.
Since c is an upper bound and none of the x's above are in S.
then $c-\delta$ is also an upper bound, This contradicts c being the least upper bound.

Therefore $f(c) = k$.

Prop 4.9: Let $f : [0,1] \to [0,1]$ be continuous, then there is $c \in [0,1]$ such that $f(c) = c$.



Thus there is $c \in [0,1]$ such that $f(c) = c$.

Proof: Let $g(x) = f(x) - x$ (then $g(x) = 0$ implies $f(x) = x$.)

Try to apply the intermediate value theorem to g.

$g(0) = f(0) - 0 = f(0) \in [0,1]$ so $g(0) \geq 0$.

$g(1) = f(1) - 1 \in [-1, 0]$ so $g(1) \leq 0$.

If $g(0)=0$ or $g(1)=0$ then we already have a fixed point. Therefore, assume $g(0)>0$ and $g(1)<0$.

By the intermediate value theorem there is $c \in (0,1)$ such that $g(c)=0$. ☑

Uniform continuity

First remember continuities.

① a function $f: D \to \mathbb{R}$ is continuous on D if it is continuous at x for all $x \in D$.

Thus, (pointwise) continuity of f on D is.

$\forall x \in D, \forall \varepsilon > 0\ \exists \delta > 0$ st. $y \in D$ and $|x-y| < \delta$ implies $|f(x)-f(y)| < \varepsilon$.

What is uniform continuity?

Uniform continuity of f on D is

$\forall \varepsilon > 0\ \exists \delta > 0$ st. $x, y \in D$ and $|x-y| < \delta$ implies $|f(x)-f(y)| < \varepsilon$.

What are the consequences of moving the $\forall x \in D$?

Uniform continuity is a stronger condition.

Thus Uniform continuity implies continuity.

Theorem 4.4 A continuous function on a closed interval is uniformly continuous there.

If $f: [a,b] \to \mathbb{R}$ is continuous then it's uniformly continuous.