# Section 4.6

## One sided limits:

Let $I \subseteq \mathbb{R}$ be an interval or a ray or $\mathbb{R}$ itself.

open, closed
half open ...

$I = (a, b), [a, b], (a, b], [a, b), (a, \infty), [a, \infty), (-\infty, b), (-\infty, b]$ or $\mathbb{R}$

Suppose $f : I \to \mathbb{R}$

define the restriction of $f$ to $D$

Let $D \subseteq I$ then $f\big|_D : D \to \mathbb{R}$

$$f\big|_D(x) = f(x) \quad \text{for } x \in D.$$

Let $c \in \mathbb{R}$ and suppose there is $b > c$ such that $(c, b) \subseteq I$

Define the limit from the right

$$f(c^+) = \lim_{x \downarrow c} f(x) = \lim_{x \to c} f\big|_{(c,b)}(x).$$

suppose there is $d < c$ such that $(d, c) \subseteq I$

Define the limit from the left

$$f(c^-) = \lim_{x \uparrow c} f(x) = \lim_{x \to c} f\big|_{(d,c)}(x).$$

Remark

$$f(c^+) = \lim_{x \to c} f\big|_{I \cap (c,\infty)}(x) \qquad f(c^-) = \lim_{x \to c} f\big|_{I \cap (-\infty,c)}(x)$$

**Definition 4.8** Let $c$ be in $I$ and suppose that $f$ is discontinuous at $c$. If $c$ is an interior point of $I$, then $f$ has a discontinuity of the *first kind*, or a *simple discontinuity*, at $c$ if both $f(c+)$ and $f(c-)$ exist in $\mathbb{R}$; otherwise, $f$ has a discontinuity of the *second kind* at $c$. If $c$ is an endpoint of $I$, then $f$ has a discontinuity of the *first kind* at $c$ if the appropriate one-sided limit of $f$ at $c$ exists in $\mathbb{R}$; otherwise, $f$ has a discontinuity of the *second kind* at $c$.

$f$ has a discontinuity of the first kind at $c$ means $f(c^+)$ and $f(c^-)$ exist in $\mathbb{R}$.

Any discontinuity that is not of the first kind is of the second kind.

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Let $f: I \to \mathbb{R}$ then

(1) $f$ is monotone increasing (decreasing) means $x, y \in I$ with $x < y$ implies $f(x) \leq f(y)$ $(f(x) \geq f(y))$

(2) $f$ is strictly increasing (decreasing) means $x, y \in I$ with $x < y$ implies $f(x) < f(y)$. $(f(x) > f(y))$

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**Example 4.24** Let

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Note $f$ is not continuous at any point $c \in \mathbb{R}$.
also $f(c^+)$ does not exist. So each discontinuity is of the second kind.

**Example 4.25** Let

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

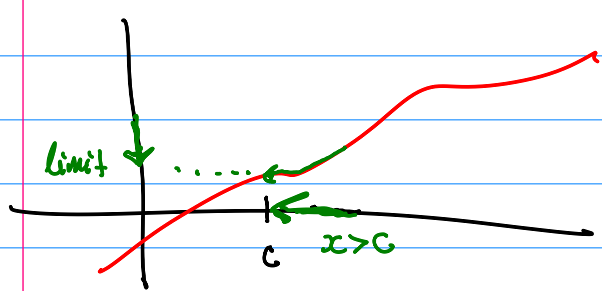
The pieces are continuous, but where they meet there is a discontinuity at $c=0$.

$$f(0^-) = \lim_{x \nearrow 0} f(x) = \lim_{x \nearrow 0} \sin \tfrac{1}{x} \quad \text{doesn't exist.}$$

So discontinuity is of second kind.

Theorem: If $f: I \to \mathbb{R}$ is monotone and $c$ is an interior point of $I$. Then $f(c^+)$ and $f(c^-)$ exist.

Assume monotone increasing.




let $\alpha = \inf\{f(x) : x \in I \cap (c, \infty)\}$

Claim $f(c^+) = \alpha$.

Note since $f$ is monotone inceasing

$x \in I \cap (c, \infty)$ implies $f(c) \leq f(x)$.

Therefore the set $\{f(x) : x \in I \cap (c, \infty)\}$ is bounded below and by the completeness axiom $\alpha \in \mathbb{R}$.

Let $\varepsilon > 0$. Since $\alpha + \varepsilon > \alpha$ and $\alpha$ is the greatest lower bound, then $\alpha + \varepsilon$ is not a lower bound. Thus, there is $b \in I \cap (c, \infty)$ such that $f(b) < \alpha + \varepsilon$.

Choose $\delta = b - c > 0$. Claim $x \in I \cap (c, \infty)$ and $|x - c| < \delta$ implies $|f(x) - \alpha| < \varepsilon$. Check this

$|x - c| < \delta$ implies $x - c < \delta = b - c$ so $x < b$.

Thus $c < x < b$. Since $f$ is monotone,

Thus $\alpha \leq f(x) \leq f(b)$ since $\alpha$ is a lower bound.

It follows

$$f(x) - \alpha \leq f(b) - \alpha < \alpha + \varepsilon - \alpha = \varepsilon$$

Therefore $|f(x) - \alpha| < \varepsilon$.

Therefore $f(c^+) = \lim_{x \searrow c} f(x) = \alpha$ and it exists

Similar argument for $f(c-)$.

Corollary: Monotone functions have no discontinuities of the second kind.

**Theorem 4.9** The set of discontinuities of a monotone function is countable.

For next time...