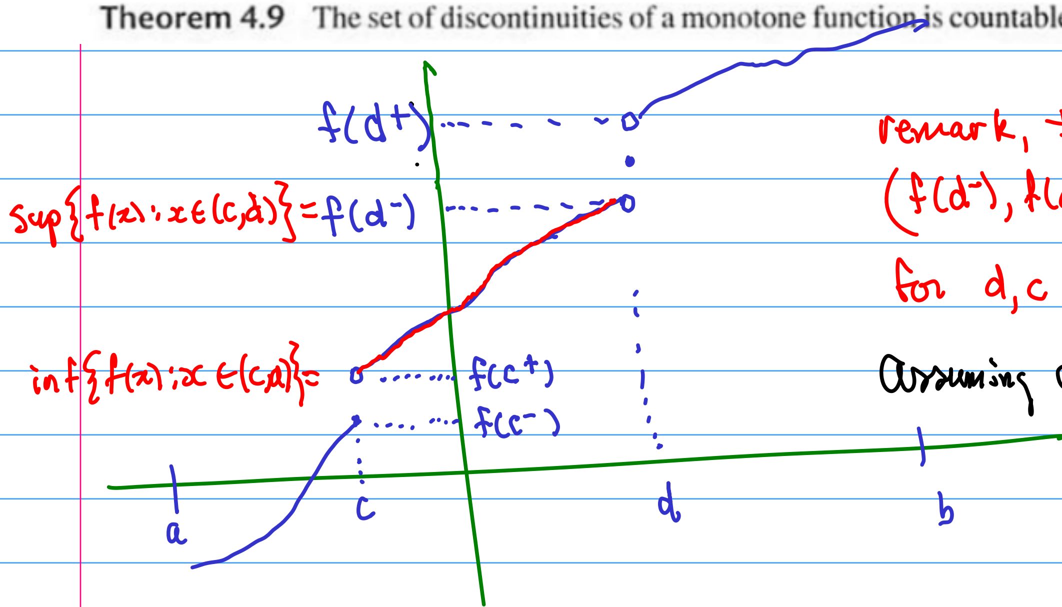
Theorem 4.9 The set of discontinuities of a monotone function is countable.



remark, the interval $(f(d^-), f(d^+)) \cap (f(c^-), f(c^+)) = \emptyset$ for $d, c \in D$.

Assuming $c < d$ we need to show $f(c^+) \le f(d^-)$

Proof: assume f is monotone increasing like the picture.
At each point c of discontinuity $f(c^-) < f(c^+)$
Thus there is a rational number $r(c) \in (f(c^-), f(c^+))$
by the density of the rational numbers.

$$D = \{ c : c \text{ is a discontinuity of } f \}$$

$$r : D \to \mathbb{Q}$$

Claim r is 1-to-1 (that is it's a bijection onto $r(D)$).

Assuming $c < d$ we need to show $f(c^+) \le f(d^-)$

$$f(c^+) = \inf \{ f(x) : x \in (c,d) \}$$

$$f(d^-) = \sup \{ f(x) : x \in (c,d) \}$$

Since $\inf\{f(x) : x \in (c,d)\} \le \sup\{f(x) : x\in(c,d)\}$
then $f(c^+) \le f(d^-)$.

Therefore the intervals $(f(d^-), f(d^+)) \cap (f(c^-), f(c^+)) = \emptyset$ for $c \neq d$.

So $r(c) \neq r(d)$ and r is 1-to-1.

$$r : D \to \mathbb{Q}$$

So r is a bijection from D onto $r(D) \subseteq \mathbb{Q}$

countable

so a subset is countable

Therefore $D \sim r(D)$ implies D is countable.

Chapter 5: Derivatives.

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

a function $f : I \to \mathbb{R}$ is differentiable at $c \in I$

if the limit $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists.

Then we define $f'(c)$ to be that limit.

Proposition 5.1 Let $f, g : I \to \mathbb{R}$ both be differentiable at c in I. Let a be in $\mathbb{R}$. Then $f \pm g$, fg, af, and f/g [if $g(c) \neq 0$] are differentiable at c and

1. $(f \pm g)'(c) = f'(c) \pm g'(c)$,
2. $(fg)'(c) = f(c)g'(c) + g(c)f'(c)$,
3. $(af)'(c) = af'(c)$,
4. $\left(\dfrac{f}{g}\right)'(c) = \dfrac{g(c)f'(c) - f(c)g'(c)}{g^2(c)}$.

Proof of 4 is in the book pg. 96. Let's try 2.

2. $(fg)'(c) = f(c)g'(c) + g(c)f'(c)$.

Need to show: $\lim\limits_{x \to c} \dfrac{f(x)g(x) - f(c)g(c)}{x-c}$ exists ...

$$\frac{f(x)g(x) - f(c)g(c)}{x-c} = \frac{f(x)g(x) - f(x)g(c)}{x-c} + \frac{f(x)g(c) - f(c)g(c)}{x-c}$$

Then

$$\lim_{x \to c} \frac{f(x)g(x) - f(c)g(c)}{x-c} = \lim_{x \to c} \left(\frac{f(x)g(x) - f(x)g(c)}{x-c} + \frac{f(x)g(c) - f(c)g(c)}{x-c} \right)$$

$$= \lim_{x \to c} \frac{f(x)g(x) - f(x)g(c)}{x-c} + \lim_{x \to c} \frac{f(x)g(c) - f(c)g(c)}{x-c}$$

$$= \lim_{x \to c} f(x) \underbrace{\lim_{x \to c} \frac{g(x) - g(c)}{x-c}}_{\text{derivatives}} + \underbrace{\lim_{x \to c} \frac{f(x) - f(c)}{x-c}}_{\text{derivative}} \lim_{x \to c} \underbrace{g(c)}_{\text{const.}}$$

$$= \left(\lim_{x \to c} f(x) \right) g'(c) + f'(c)\, g(c).$$

What is $\left(\lim\limits_{x \to c} f(x) \right)$? Claim f is continuous at c.

$$\lim_{x \to c} f(x) = \lim_{x \to c} \left(f(x) - f(c) \right) + \lim_{x \to c} f(c)$$

$$\lim_{x\to c}\left(f(x)-f(c)\right) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}\cdot(x-c)$$

$$= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \quad \lim_{x\to c}(x-c)$$

$$= f'(c)\,(c-c) = 0.$$

$$\lim_{x\to c} f(x) = 0 + \lim_{x\to c} f(c) = f(c),$$

It follows that

$$\lim_{x\to c} \frac{f(x)g(x)-f(c)g(c)}{x-c} = \left(\lim_{x\to c} f(x)\right) g'(c) + f'(c)\,g(c)$$

implies $\quad (fg)'(c) = f(c)\,g'(c) + f'(c)\,g(c).$

Theorem 5.2 (chain rule) Let I and J be intervals or rays in $\mathbb{R}$, let $f : I \to J$ and $g : J \to \mathbb{R}$, and let c be in I with f differentiable at c and g differentiable at $f(c)$. Then the composite function $g \circ f$ is differentiable at c and

$$(g \circ f)'(c) = g'(f(c))f'(c).$$

$$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx}$$ fake proof!...

Trying to compute and show it exists

$$\lim_{x\to c} \frac{g(f(x))-g(f(c))}{x-c}$$

Fake proof #2

$$\lim_{x\to c}\frac{g(f(x))-g(f(c))}{x-c} = \lim_{x\to c}\frac{g(f(x))-g(f(c))}{f(x)-f(c)}\;\frac{f(x)-f(c)}{x-c}$$

problem: what if $f(x)=f(c)$? what if f is not a 1-to-1 function?

$$= \lim_{x\to c}\frac{g(f(x))-g(f(c))}{f(x)-f(c)}$$

Let $z=f(x)$ so $z\to f(c)$ as $x\to c$

$$= \lim_{z\to f(c)}\frac{g(z)-g(f(c))}{z-f(c)}\;\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

$$= g'(f(c))\,f'(c)$$

Proof in the book (simplied a little bit).

$$d(z) = \begin{cases}\dfrac{g(z)-g(f(c))}{z-f(c)} & \text{for } z\neq f(c)\\[2ex] g'(f(c)) & \text{for } z=f(c)\end{cases}$$

Note that $d(z)$ is continuous because

$$\frac{g(f(x))-g(f(c))}{x-c} = d(f(x))\,\frac{f(x)-f(c)}{x-c}$$

this is equal because if $f(x)=f(c)$ then it is

$$0 = g'(f(c))\cdot 0 = 0.$$

$$\lim_{x \to c} \frac{g(f(x)) - g(f(c))}{x - c} = \lim_{x \to c} d(f(x)) \, \frac{f(x) - f(c)}{x - c}$$

$$= \lim_{x \to c} d(f(x)) \; \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

composition of cont. function

$$= d(f(c)) \, f'(c) = g'(f(c)) \, f'(c).$$