Example $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$

If $c \neq 0$ then

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \frac{d}{dx} x^2 \sin \frac{1}{x} \Big|_{x=c} = 2x \sin \frac{1}{x} - \cancel{x^2} \left(\cos \frac{1}{x}\right) \cancel{x^{-2}} \Big|_{x=c}$$

If $c \neq 0$ then $f'(c) = 2c \sin \frac{1}{c} - \cos \frac{1}{c}$

If $c = 0$ then

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin \frac{1}{x}}{x} = \lim_{x \to 0} x \sin \frac{1}{x} = 0$$

$$f'(x) = \begin{cases} 2c \sin \frac{1}{x} - \cos \frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Is $f'(x)$ continuous?

Note that $\lim_{x \to 0} f'(x) = \lim_{x \to 0} \left(2c \sin \frac{1}{x} - \cos \frac{1}{x} \right)$

oscillates so the limit doesn't exist.

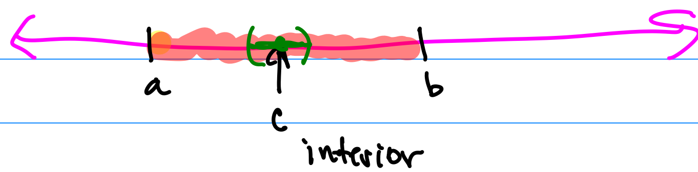
Note the discontinuity of $f'(x)$ at $x=0$ is of the second kind.

Proposition 5.2 Suppose that f has a local maximum or a local minimum at an interior point c of I. If f is differentiable at c, then $f'(c) = 0$.

Definition 5.2 Let $f : I \to \mathbb{R}$ with c in I. Then f has a *local maximum* (respectively, *local minimum*) at c if there is a neighborhood U of c such that $f(x) \le f(c)$ [respectively, $f(x) \ge f(c)$] for all x in $U \cap I$.

<u>Proof</u> of Prop 5.2

Let $c \in I$ be an interior point. (That is c is not an end point of I.)



Thus there is $\delta > 0$ such that $(c-\delta, c+\delta) \subseteq I$

Suppose c is a local minimum. Then by choosing δ smaller if needed we have $f(x) \geq f(c)$ for all $x \in (c-\delta, c+\delta)$.

numerator is ≥ 0

If $x \in (c-\delta, c)$ then $\dfrac{f(x)-f(c)}{x-c} \leq 0$

denominator is ≤ 0

Consequently $f'(c) = \lim\limits_{x \to c} \dfrac{f(x)-f(c)}{x-c} = \lim\limits_{\substack{x \to c \\ x < c}} \dfrac{f(x)-f(c)}{x-c} \leq 0$

numerator is ≥ 0

If $x \in (c, c+\delta)$ then $\dfrac{f(x)-f(c)}{x-c} \geq 0$

denominator is ≥ 0

Consequently $f'(c) = \lim\limits_{x \to c} \dfrac{f(x)-f(c)}{x-c} = \lim\limits_{\substack{x \to c \\ x > c}} \dfrac{f(x)-f(c)}{x-c} \geq 0$

Since $f'(c) \leq 0$ and $f'(c) \geq 0$ then $f'(c) = 0$.

Similar proof for when $f(c)$ is a relative maximum is in the textbook.

Theorem 5.3 If f is differentiable on I, then f' has the intermediate value property on I. That is, given a and b in I with $f'(a) \neq f'(b)$ and r between $f'(a)$ and $f'(b)$, there is a c between a and b such that $f'(c) = r$.

Note that if we assume f' is continuous, then we know all continuous functions have the intermediate value property so there is nothing more to do in this case.

Proof: Let $F(x) = f(x) - rx$. Note this function has been constructed so that $F'(x) = f'(x) - r$ and what we want is to solve for c such that $F'(c) = 0$.

If F has a relative maximum or minimum at an interior point c of I then $F'(c) = 0$ and we've found c.

Now consider the cases where both max and min are at the endpoints.

Case 1. $F(a)$ is min and $F(b)$ is max

Case 2. $F(a)$ is max and $F(b)$ is min.

For definiteness, lets assume

$a < b$ and $f'(a) < f'(b)$ done in the book ☑

or $a < b$ and $f'(a) > f'(b)$ ☑

Note we can assume $a < b$ without loss of generality by relabeling them if needed.

$F(x) = f(x) - rx$

Assume $a < b$ and $f'(a) > \boxed{r > f'(b)}$.

Case 1. $F(a)$ is min and $F(b)$ is max

$$f'(b) - r = F'(b) = \lim_{\substack{x \to b \\ x < b}} \frac{F(x) - F(b)}{x - b} \geq 0 \qquad \text{(numerator } \leq 0\text{, denominator } \leq 0\text{)}$$

thus $\boxed{f'(b) \geq r}$ contradiction... so case 1 doesn't happen...

Case 2. $F(a)$ is max and $F(b)$ is min.

No contradiction here

$$f'(b) - r = F'(b) = \lim_{\substack{x \to b \\ x < b}} \frac{F(x) - F(b)}{x - b} \leq 0 \qquad \text{(numerator } \geq 0\text{, denominator } \leq 0\text{)}$$

thus $\boxed{f'(b) \leq r}$ no contradiction.

$\therefore$

$$f'(a) - r = F'(a) = \lim_{\substack{x \to a \\ x > a}} \frac{F(x) - F(a)}{x - a} \leq 0 \qquad \text{(numerator } \leq 0\text{, denominator } \geq 0\text{)}$$

thus $f'(a) \leq r$

but $f'(a) > r$ by hy hypothesis

contradiction... so case 2 doesn't happen...

Corollary 5.1 If f is differentiable on I, then all discontinuities of f' are of the second kind.

Proof Referring to Section 4.6, if f' has a discontinuity of the first kind at a point c in I, then f' has a jump discontinuity at c. Thus f' cannot satisfy the intermediate value property in an interval around c. ■

Generalized mean value theorem next time

$f:[a,b]\to\mathbb{R}$ $\quad g:[a,b]\to\mathbb{R}$ continuous and differentiable on (a,b).