$P_n(x_0) = f(x_0),\ P_n'(x_0) = f'(x_0), \ldots,\ P_n^{(n)}(x_0) = f^{(n)}(x_0)$

Recall

n+1 differentiable function

P_n is a polynomial of degree n.

$$g(t) = f(t) - P_n(t) - M(t - x_0)^{n+1}$$

$$M = \boxed{\frac{f(x) - P_n(x)}{(x - x_0)^{n+1}}}$$

There is a c between x and x_0 such that $g^{(n+1)}(c) = 0$.

$$g^{(n+1)}(t) = f^{(n+1)}(t) - 0 - (n+1)!\, M$$

$$g^{(n+1)}(c) = f^{(n+1)}(c) - 0 - (n+1)!\, M = 0$$

Therefore $\quad M = \dfrac{f^{(n+1)}(c)}{(n+1)!} = \boxed{\dfrac{f(x) - P_n(x)}{(x - x_0)^{n+1}}}$

$$R_n(x) = \boxed{f(x)} - \boxed{P_n(x)} = \boxed{\frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}}$$ for some c between x and x_0

Theorem 5.6 (Taylor's Theorem) Suppose that $f : [a, b] \to \mathbb{R}$, n is a positive integer, $f^{(n)}$ is continuous on $[a, b]$, and $f^{(n)}$ is differentiable on (a, b). For $x \neq x_0$ in $[a, b]$, there is a c between x and x_0 such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

Quiz 5 will be next Friday over homework that I will post tonight...

42 May 06 *** Review

*** Prep Day May 08
*** Final exam Wednesday, May 15 from 8:00-10:00am in PE103

Chapter 6: Riemann Integration...

Definition 6.1: A partition of the interval $[a,b]$ is a set of points $\{x_0, x_1, \ldots, x_n\} \subseteq [a,b]$ with $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

$$\mathcal{P} = \mathcal{P}[a,b] = \{ P : P \text{ is a partition of } [a,b] \}$$

Given a partition $P \in \mathcal{P}[a,b]$ and $f : [a,b] \to \mathbb{R}$

Let $M_i = \sup \{ f(x) : x \in [x_{i-1}, x_i] \}$

$m_i = \inf \{ f(x) : x \in [x_{i-1}, x_i] \}$

length of this interval.

The upper sum.

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i \quad \text{where } \Delta x_i = x_i - x_{i-1}$$

The lower sum

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i \quad \text{where } \Delta x_i = x_i - x_{i-1}$$

Remark $U(P,f) \geq L(P,f)$

The upper Riemann integral

$$\overline{\int_a^b} f = \inf \{ U(P,f) : P \in \mathcal{P}[a,b] \}$$

The lower Riemann integral

$$\underline{\int_a^b} f = \sup \{ L(P,f) : P \in \mathcal{P}[a,b] \}$$

Substitute definitions into one another to see

$$\overline{\int_a^b} f = \inf \left\{ \sum_{i=1}^{n} \sup \{ f(x) : x \in [x_{i-1}, x_i] \} \Delta x_i : \{x_0, x_1, \dots, x_n\} \in \mathcal{P}[a,b] \right\}$$

"Smallest of the large things"

$$\underline{\int_a^b} f = \sup \left\{ \sum_{i=1}^{n} \inf \{ f(x) : x \in [x_{i-1}, x_i] \} \Delta x_i : \{x_0, x_1, \dots, x_n\} \in \mathcal{P}[a,b] \right\}$$

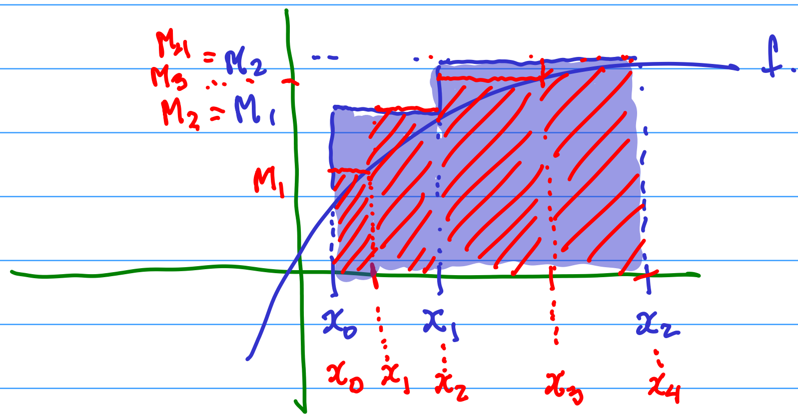
"largest of the small things"

Definition 6.4 A bounded function f is *Riemann integrable* on $[a, b]$ if $\underline{\int_a^b} f = \overline{\int_a^b} f$, and this common value is the *Riemann integral* of f over $[a, b]$.

$$\mathcal{R} = \left\{ f : [a,b] \to \mathbb{R} \text{ such that } \underline{\int_a^b} f = \overline{\int_a^b} f \right\}$$

Definition: If $P, P^* \in \mathcal{P}[a,b]$ with $P \subseteq P^*$ then we say P^* is a refinement of P.

If P^* is a refinement of P then



$P = \{x_0, x_1, x_2\}$

$P^* = \{x_0, x_1, x_2, x_3, x_4\}$

What are approximations from above

$U(P,f) = \sum_{i=1}^{2} M_i \Delta x_i$ $\quad$ $U(P^*,f) = \sum_{i=1}^{4} M_i \Delta x_i$

Note that $\quad U(P^*, f) \leq U(P,f)$

also $\quad L(P^*, f) \geq L(P,f)$

Proof We prove the first inequality, leaving the second one for the reader. Let $P = \{x_i\}_{i=0}^{n}$ be in $\mathcal{P}$ and suppose first that P^* contains just one more point than P. So $P^* = P \cup \{z\}$, where $x_{k-1} < z < x_k$ for some k in $\{1, 2, \ldots, n\}$. Then

$$L(P^*, f) = \sum_{i \neq k} m_i \Delta x_i + m'(z - x_{k-1}) + m''(x_k - z),$$

where

$$m' = \inf\{f(x) : x_{k-1} \leq x \leq z\}$$

and

$$m'' = \inf\{f(x) : z \leq x \leq x_k\}.$$

Since $m_k = \inf\{f(x) : x_{k-1} \le x \le x_k\}$, $m_k \le m'$ and $m_k \le m''$. Thus,

$$L(P^*, f) - L(P, f) = m'(z - x_{k-1}) + m''(x_k - z) - m_k(x_k - x_{k-1})$$
$$= (m' - m_k)(z - x_{k-1}) + (m'' - m_k)(x_k - z) \ge 0.$$

If P^* contains j more points than P, we repeat the reasoning above j times to arrive at $L(P, f) \le L(P^*, f)$. ■

read this and try to relate it
to the picture we just drew.

Proposition 6.2 If P_1 and P_2 are in $\mathcal{P}$, then

$$L(P_1, f) \le U(P_2, f).$$

proof: Let $P^* = P_1 \cup P_2$.

Then $P_1 \subseteq P^*$ so P^* is a refinement of P_1 but
also $P_2 \subseteq P^*$ so P^* is a refinement of P_2.

$$L(P_1, f) \le L(P^*, f) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} \inf\{f(x) : x \in [x_{i-1}, x_i]\} \Delta x_i$$

$$\le \sum_{i=1}^{n} \sup\{f(x) : x \in [x_{i-1}, x_i]\} \Delta x_i = U(P^*, f) \le U(P_2, f).$$