Recall ...

Proposition 6.2 If P_1 and P_2 are in $\mathcal{P}$, then

$$L(P_1, f) \leq U(P_2, f).$$

We want to show

Theorem 6.1 $\underline{\int_a^b} f \leq \overline{\int_a^b} f.$

This follows almost immediately from Prop 6.2. Since

$$L(P_1, f) \leq U(P_2, f) \qquad \text{for all} \quad P_1, P_2 \in \mathcal{P}[a,b]$$

Take limits (actually infimum) with P_2 to obtain

doesn't depend on P_2 so unaffected

$$L(P_1, f) \leq \inf\{U(P, f) : P \in \mathcal{P}[a,b]\}$$

Now take limits (actually supremum) with P_1

unchanged since $\inf\{U(P, f) : P \in \mathcal{P}[a,b]\}$ is const. w.r.t. P_1.

$$\sup\{L(P, f) : P \in \mathcal{P}[a,b]\} \leq \inf\{U(P, f) : P \in \mathcal{P}[a,b]\}$$

Since $\underline{\int_a^b} f = \sup\{L(P, f) : P \in \mathcal{P}[a,b]\}$

and $\overline{\int_a^b} f = \inf\{U(P, f) : P \in \mathcal{P}[a,b]\}$

It follows that $\underline{\int_a^b} f \leq \overline{\int_a^b} f$.

Example: Let $f:[0,1]\to\mathbb{R}$ be defined as $f(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q}\\ 0 & \text{if } x\notin\mathbb{Q}\end{cases}$

Given a partition $P=\{x_0,x_1,\ldots,x_n\}\in\mathcal{P}[a,b]$ then

$$M_i=\sup\{f(x): x\in[x_{i-1},x_i]\}=1$$

Since $\mathbb{Q}$ is dense and so there are rational points in any interval (of positive length).

$$m_i=\inf\{f(x): x\in[x_{i-1},x_i]\}=0$$

Since the irrationals $\mathbb{R}\setminus\mathbb{Q}$ are also dense.

It follows

$$U(P,f)=\sum_{i=1}^{n} M_i\Delta x_i=\sum_{i=1}^{n}\Delta x_i=\text{length of }[0,1]=1.$$

and

$$L(P,f)=\sum_{i=1}^{n} m_i\Delta x_i=0$$

Consequently $\overline{\int_0^1} f=1$ and $\underline{\int_0^1} f=0.$

Therefore

$$f(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q}\\ 0 & \text{if } x\notin\mathbb{Q}\end{cases}$$

is not Riemann integrable so $f\notin\mathcal{R}[a,b]$.

Theorem 6.2 A bounded function f is in $\mathcal{R}[a,b]$ if and only if for every $\varepsilon > 0$ there is a partition P of $[a,b]$ such that $U(P,f) - L(P,f) < \varepsilon$.

Suppose $f:[a,b] \to \mathbb{R}$ is bounded

" $\Longrightarrow$ " Let $f \in \mathcal{R}[a,b]$. Claim $\forall \varepsilon > 0\ \exists P \in \mathcal{P}[a,b]$ s.t $U(P,f) - L(P,f) < \varepsilon$.

Let $\varepsilon > 0$ be arbitrary. By definition

$$\underline{\int_a^b} f = \sup \{ L(P,f) : P \in \mathcal{P}[a,b] \}.$$

least upper bound

Since $\underline{\int_a^b} f$ is the least upper bound, then $\underline{\int_a^b} f - \frac{\varepsilon}{4}$ is not an upper bound. Consequently there is $P_1 \in \mathcal{P}[a,b]$ such that

$$L(P_1, f) > \underline{\int_a^b} f - \varepsilon/4$$

equivalently $-L(P_1, f) < -\underline{\int_a^b} f + \varepsilon/4$

Similarly

$$\overline{\int_a^b} f = \inf \{ U(P,f) : P \in \mathcal{P}[a,b] \}.$$

greatest lower bound

Since $\overline{\int_a^b} f$ is the greatest lower bound, then $\overline{\int_a^b} f + \frac{\varepsilon}{4}$ is not an lower bound. Consequently there is $P_2 \in \mathcal{P}[a,b]$ such that

$$U(P_2, f) < \overline{\int_a^b} f + \varepsilon/4$$

Let $P = P_1 \cup P_2$. Then P is a refinement of P_1 and P_2 so

$$U(P,f) \leq U(P_2,f) \quad \text{and} \quad L(P,f) \geq L(P_1,f).$$

$-L(P,f) \leq -L(P_1,f)$

Now

$$U(P,f) - L(P,f) \leq U(P_2,f) - L(P_1,f)$$

$$< \cancel{\overline{\int_a^b} f} + \varepsilon/4 \; \cancel{- \underline{\int_a^b} f} + \varepsilon/4 = \frac{\varepsilon}{2} < \varepsilon.$$

Note that one could have used $\varepsilon/2$ in the original argument and still obtain ε in the end.

" $\Leftarrow$ "

$\forall$ $\varepsilon > 0$ there is a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \varepsilon$.

Need to show $f \in \mathcal{R}[a,b]$.

by definition

$$-\underline{\int_a^b} f \leq -L(P,f)$$

$$\underline{\int_a^b} f = \sup\{L(P,f) : P \in \mathcal{P}[a,b]\} \geq L(P,f)$$

$$\overline{\int_a^b} f = \inf\{U(P,f) : P \in \mathcal{P}[a,b]\} \leq U(P,f)$$

Therefore

$$0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f \leq U(P,f) - L(P,f) < \varepsilon.$$

Since ε is arbitrary then $\overline{\int_a^b} f = \underline{\int_a^b} f$

Therefore $f \in \mathcal{R}[a,b]$.

6.2 Riemann Sums:

Definition $P=\{x_0,x_1,\dots,x_n\}\in\mathcal{P}[a,b]$.

Then $\|P\|=\max\{\Delta x_i : i=1,\dots,n\}$

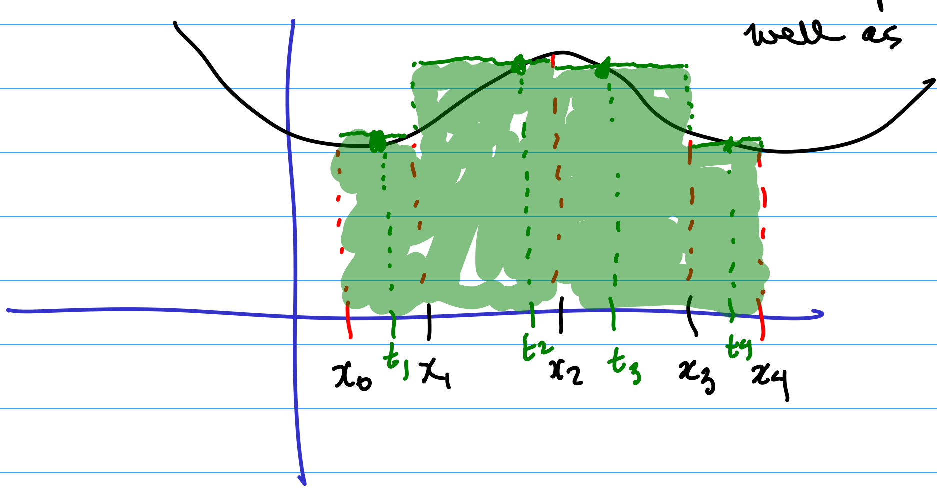
Definition

$$S(P,f)=\sum_{i=1}^{n} f(t_i)\Delta x_i \quad \text{where} \quad t_i\in[x_{i-1},x_i].$$

Remark, there are infinitely many choices for the t_i and

the Riemann sum $S(P,f)$ depends on the choice of the t_i

The numerical value of this sum depends on the t_i as well as P.



Define the limit of Riemann sums

$$\lim_{\|P\| \to 0} S(P,f) = I$$

means

$\forall \varepsilon > 0 \; \exists \delta > 0$ s.t $P \in \mathcal{P}[a,b]$ with $\|P\| < \delta$ implies $|S(P,f) - I| < \varepsilon$ for all possible choices of $t_i \in [x_{i-1}, x_i]$.

Theorem: $f : [a,b] \to \mathbb{R}$ bounded,

Then $\lim_{\|P\| \to 0} f$ exists if and only if $f \in \mathcal{R}[a,b]$.