Theorem 6.3 A bounded function f is in $\mathcal{R}[a,b]$ if and only if $\lim_{\|P\|\to 0} S(P,f)$ exists in $\mathbb{R}$, and then $\int_a^b f = \lim_{\|P\|\to 0} S(P,f)$.

"$\Rightarrow$" Done on blackboard.

"$\Leftarrow$" Suppose $\lim_{\|P\|\to 0} S(P,f)$ exists. Claim $f \in \mathcal{R}[a,b]$.

want to use

Theorem 6.2 A bounded function f is in $\mathcal{R}[a,b]$ if and only if for every $\varepsilon > 0$ there is a partition P of $[a,b]$ such that $U(P,f) - L(P,f) < \varepsilon$.

so need to find a partition so $U(P,f) - L(P,f) < \varepsilon$.

Let $\varepsilon > 0$. By hypothesis there is $I \in \mathbb{R}$ such that for $\varepsilon_1 = \boxed{\varepsilon/4} > 0$ there is $\delta_1 > 0$ such that $P \in \mathcal{P}[a,b]$ with $\|P\| < \delta_1$ implies

$$|S(P,f) - I| < \varepsilon_1 \text{ for every choice of } t_i \in [x_{i-1}, x_i]$$

Recall

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i \quad \text{where } M_i = \sup\{f(t) : t \in [x_{i-1}, x_i]\}.$$

Let $\varepsilon_2 = \boxed{\dfrac{\varepsilon}{4(b-a)}} > 0$

Since M_i is the least upper bound then $M_i - \varepsilon_2$ is not an upper bound. Thus there is $t_i \in [x_{i-1}, x_i]$ such that

$$f(t_i) > M_i - \varepsilon_2 \qquad \text{or} \qquad M_i < f(t_i) + \varepsilon_2$$

Do this for every i.

Recall

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i \quad \text{where } m_i := \inf \{ f(t) : t \in [x_{i-1}, x_i] \}.$$

So $m_i + \varepsilon_2$ is not a lower bound. So there is

$t_i' \in [x_{i-1}, x_i]$ such that

$$f(t_i') < m_i + \varepsilon_2 \qquad m_i > f(t_i') - \varepsilon_2$$

Now

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i \leq \sum_{i=1}^{n} (f(t_i) + \varepsilon_2) \Delta x_i$$

$$= \sum_{i=1}^{n} f(t_i) \Delta x_i + \varepsilon_2 \sum_{i=1}^{n} \Delta x_i$$

$$= \sum_{i=1}^{n} f(t_i) \Delta x_i + \varepsilon_2 (b-a)$$

also

$$L(P,f) \geq \underbrace{\sum_{i=1}^{n} f(t_i') \Delta x_i}_{\text{Riemann sums.}} - \varepsilon_2 (b-a)$$

Since

$|S(P,f) - I| < \varepsilon_1$ for every choice of $t_i \in [x_{i-1}, x_i]$

Then

$$\left| \sum_{i=1}^{n} f(t_i') \Delta x_i - I \right| < \varepsilon_1$$

or $-\varepsilon_1 < \sum_{i=1}^{n} f(t_i') \Delta x_i - I < \varepsilon_1$ so $-\sum_{i=1}^{n} f(t_i') \Delta x_i < -I + \varepsilon_1$

also $-\varepsilon < \boxed{\sum_{i=1}^{n} f(t_i)\Delta x_i - I < \varepsilon_1}$ so $\sum_{i=1}^{n} f(t_i)\Delta x_i < \varepsilon_1 + I$

What do we have?,

$$L(P,f) \geq \sum_{i=1}^{n} f(t_i')\Delta x_i - \varepsilon_2(b-a)$$

$$U(P,f) \leq \sum_{i=1}^{n} f(t_i)\Delta x_i + \varepsilon_2(b-a)$$

$$U(P,f) - L(P,f) \leq \sum_{i=1}^{n} f(t_i)\Delta x_i + \varepsilon_2(b-a) - \left(\sum_{i=1}^{n} f(t_i')\Delta x_i - \varepsilon_2(b-a)\right)$$

$$= \sum_{i=1}^{n} f(t_i)\Delta x_i - \sum_{i=1}^{n} f(t_i')\Delta x_i + 2\varepsilon_2(b-a)$$

$$< \varepsilon_1 + \cancel{I} - \cancel{I} + \varepsilon_1 + 2\varepsilon_2(b-a)$$

recall

$$-\sum_{i=1}^{n} f(t_i')\Delta x_i < -I + \varepsilon_1$$

$$\sum_{i=1}^{n} f(t_i)\Delta x_i < \varepsilon_1 + I$$

$$= 2\varepsilon_1 + 2\varepsilon_2(b-a)$$

$$= \cancel{2}\frac{\varepsilon}{\cancel{4}_{2}} + \cancel{2}\left(\varepsilon / \cancel{4}_{2}\cancel{(b-a)}\right)\cancel{(b-a)}$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \square$$

Proposition 6.3 Let f and g be in $\mathcal{R}[a, b]$ and let c be in $\mathbb{R}$. Then $f \pm g$ and cf are in $\mathcal{R}[a, b]$, and

$$\int_a^b (f \pm g) = \int_a^b f \pm \int_a^b g$$

and

$$\int_a^b cf = c\int_a^b f.$$

linearity ...

Proposition 6.4 Let f and g be in $\mathcal{R}[a,b]$, with $f(x) \le g(x)$ for all x in $[a,b]$. Then $\int_a^b f \le \int_a^b g$.

Proposition 6.5 Let f be in $\mathcal{R}[a,b]$ and let $a < c < b$. Then f is in $\mathcal{R}[a,c]$, f is in $\mathcal{R}[c,b]$, and $\int_a^b f = \int_a^c f + \int_c^b f$.

<u>Proof of 6.4</u> Let $h(x) = g(x) - f(x)$. Then $h(x) \geq 0$

$$L(P,h) = \sum_{i=1}^{n} m_i \Delta x_i \quad \text{where } m_i = \inf\{h(t) : t \in [x_{i-1}, x_i]\} \geq 0$$

Thus $L(P,h) \geq 0$

Thus $$\underline{\int_a^b} h = \sup\{L(P,h) : P \in \mathcal{P}[a,b]\} \geq 0$$

Claim $h \in \mathcal{R}[a,b]$. Why. By prop 6.3.

Thus $$\underline{\int_a^b} h = \overline{\int_a^b} h = \int_a^b h.$$

So $$0 \leq \int_a^b h = \int_a^b (g-f) = \int_a^b g - \int_a^b f$$

So $$\int_a^b f \leq \int_a^b g.$$