Notation

$\mathbb{R}$ is the set of real numbers.

$\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of positive integers (or natural numbers).

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the set of integers.

$\mathbb{Q} = \{m/n : m, n \in \mathbb{Z}, n \neq 0\} = \{m/n : m \in \mathbb{Z}, n \in \mathbb{N}\}$ is the set of rational numbers.

$\mathbb{R} \setminus \mathbb{Q}$, the complement of $\mathbb{Q}$ in $\mathbb{R}$, is the set of irrational numbers.

Idea: Base the theory of Calculus of a small set of axioms.

Field axioms of $\mathbb{R}$

Axiom 2.1

$$a + b = b + a$$
$$a \cdot b = b \cdot a \qquad \text{(commutative laws)}$$

Axiom 2.2 For all c in $\mathbb{R}$,

$$a + (b + c) = (a + b) + c$$
$$a \cdot (b \cdot c) = (a \cdot b) \cdot c \qquad \text{(associative laws)}$$

Axiom 2.3 For all c in $\mathbb{R}$,

$$a \cdot (b + c) = a \cdot b + a \cdot c \qquad \text{(distributive law)}$$

Axiom 2.4 There exist distinct real numbers 0 and 1 such that for all a in $\mathbb{R}$,

$$a + 0 = a$$
$$a \cdot 1 = a \qquad \text{(identity elements)}$$

Axiom 2.5 For each a in $\mathbb{R}$, there is an element $-a$ in $\mathbb{R}$ such that

$$a + (-a) = 0$$

and for each b in $\mathbb{R}$, $b \neq 0$, there is an element $b^{-1} = 1/b$ in $\mathbb{R}$ such that

$$b \cdot \frac{1}{b} = 1. \qquad \text{(inverse elements)}$$

Order axioms of $\mathbb{R}$

Axiom 2.6 For all a and b in $\mathbb{R}$, exactly one of the following holds:

$a = b,\ a < b,\ b < a$ (trichotomy).

Axiom 2.7 For all a, b, and c in $\mathbb{R}$, if $a < b$, then $a + c < b + c$.

Axiom 2.8 For all a, b, and c in $\mathbb{R}$, if $a < b$ and $0 < c$, then $ac < bc$.

Axiom 2.9 For all a, b, and c in $\mathbb{R}$, if $a < b$ and $b < c$, then $a < c$ (transitivity).

Axiom 2.10 The positive integers are well-ordered.

a set F is well ordered if every non-empty subset of $A \subseteq F$ has a least element. Thus if $A \subseteq F$ and $A \neq \emptyset$ then there is $a_0 \in A$ s.t. $a_0 \leq a$ for all $a \in A$.

Theorem 1.3

$\mathbb{N}$ is well ordered $\Longleftrightarrow$ the principle of mathematical induction holds

"$\Rightarrow$" and then "$\Leftarrow$" to show equivalent but since we'll assume $\mathbb{N}$ is well ordered as an axiom in Chapter 2, I'll only present the direction "$\Rightarrow$" here..

Proof.

Suppose $\mathbb{N}$ is well ordered. Need to show that

if ① $p(1)$ is true

and ② $p(n) \Rightarrow p(n+1)$ for all $n \in \mathbb{N}$

then $p(n)$ is true for all $n \in \mathbb{N}$.

For contradiction, suppose the principle of mathematical induction doesn't hold true. Thus there is a $p(n)$ which satisfies ① and ② but for which $p(n)$ is NOT TRUE for all $n \in \mathbb{N}$.

Define $A = \{n : p(n) \text{ is false}\}$. Then $A \neq \emptyset$.

Since $A \subseteq \mathbb{N}$ and $\mathbb{N}$ is well ordered then there exists $n_0 \in A$ st. $n_0 \leq n$ for all $n \in A$.

Since $p(1)$ is true. Then $n_0 > 1$. because $1 \notin A$.

It follows $n_0 - 1 > 0$. which means $n_0 - 1 \in \mathbb{N}$.

Claim $p(n_0-1)$ is true. If not then $n_0 - 1 \in A$ but then $n_0 \leq n$ for all $n \in A$ implies $n_0 \leq n_0 - 1$ which is false. Thus $p(n_0-1)$ is true.

by ② $p(n_0-1) \Rightarrow p(n_0)$ so $p(n_0)$ is true.

That implies $n_0 \notin A$. This contradicts $n_0 \in A$

Decimal representation of $\mathbb{R}$. Shown that $\mathbb{R}$ actually exists... $x \in \mathbb{R}$ is given by

$$x = n_0 . a_1 a_2 a_3 a_4 \ldots$$

where $n \in \mathbb{Z}$ and $a_i \in \{0, 1, \ldots, 9\}$.

From an infinite series point of view.

$$x = n + \sum_{i=1}^{\infty} \frac{1}{10^i} a_i$$

to recall..

Note we haven't discussed limits so don't think about where this converges yet.

Note a fraction can be converted to a repeating decimal using division.

The other way... Example...

$3.61212\overline{12} = 3.6\overline{12}$ ← repeating parts

$$3 + \frac{6}{10} + \left(\frac{1}{10^2} + \frac{2}{10^3}\right) + \left(\frac{1}{10^4} + \frac{2}{10^4}\right) + \cdots$$

$$= 3 + \frac{6}{10} + \frac{1}{10^2}\left(\left(1 + \frac{2}{10}\right) + \frac{1}{10^2}\left(1 + \frac{2}{10}\right) + \cdots\right)$$

$$= 3 + \frac{6}{10} + \frac{1}{10^2}\left(1 + \frac{2}{10}\right)\underbrace{\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \cdots\right)}_{\text{geometric series}}$$

$$S = 1 + \frac{1}{100} + \left(\frac{1}{100}\right)^2 + \left(\frac{1}{100}\right)^3 + \cdots$$

$$\frac{1}{100} S = \left(1 + \frac{1}{100} + \left(\frac{1}{100}\right)^2 + \left(\frac{1}{100}\right)^3 + \cdots\right)\frac{1}{100}$$

Subtract

$$S = 1 + \cancel{\frac{1}{100}} + \cancel{\left(\frac{1}{100}\right)^2} + \cancel{\left(\frac{1}{100}\right)^3} + \cancel{\cdots}$$

$$\frac{1}{100}S = \cancel{\frac{1}{100}} + \cancel{\left(\frac{1}{100}\right)^2} + \cancel{\left(\frac{1}{100}\right)^3} + \cancel{\cdots}$$

$$S - \frac{1}{100}S = 1 \quad \text{so} \quad S = \frac{1}{1-\frac{1}{100}} = \frac{100}{99}$$

$$3.6\overline{12} = \underbrace{3 + \frac{6}{10} + \frac{1}{10^2}\left(1+\frac{2}{10}\right)\frac{100}{99}}_{\text{this is a fraction}}$$

Finish reading Section 2.1 about $|x|$

$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

and prove the properties of $|x|$ using the definition. For example.

$$|a||b| = |ab|$$

or

$$|a+b| \leq |a| + |b|$$

Proofs follow the form in cases

Case $a \geq 0$ and $b \geq 0$

Case $a < 0$ and $b \geq 0$

Case $a \geq 0$ and $b < 0$

Case $a < 0$ and $b < 0$.