Theorem 2.2 $\mathbb{Q}$ is dense in $\mathbb{R}$.

What does dense mean?

Definition: $A \subseteq \mathbb{R}$ is dense in $\mathbb{R}$ if between any two real numbers there exists an element of A.
equivalently,

$\forall x, y \in \mathbb{R}$ s.t. $x<y$ then $A \cap (x,y) \neq \emptyset$.

Proof (of $\mathbb{Q} \subseteq \mathbb{R}$ being dense)

Let $x, y \in \mathbb{R}$ s.t. $x<y$. Claim $\mathbb{Q} \cap (x,y) \neq \emptyset$.

✓ Case $x<0$ and $0<y$: Then $0 \in \mathbb{Q}$ and $0 \in (x,y)$ implies $\mathbb{Q} \cap (x,y) \neq \emptyset$.

✓ Case $x=0$ and $0<y$: Since $\frac{1}{y} > 0$ and $\mathbb{N}$ is unbounded then there is $n \in \mathbb{N}$ such that $\frac{1}{y} < n$.
Therefore $\frac{1}{n} < y$.
Thus $\frac{1}{n} \in \mathbb{Q}$ and $\frac{1}{n} \in (0,y)$ implies $\mathbb{Q} \cap (x,y) \neq 0$.

✓ Case $0<x$ and $x<y$: Then $y-x>0$ so by Archimedean principle there is n such that $\frac{1}{n} < y-x$.

Need to find a fraction between x and y.

Let $A = \{ \frac{m}{n} : \frac{m}{n} < y \}$.

Claim $A \neq \emptyset$. $\frac{1}{n} < y-x$ so $\frac{1}{n} < y$ since $x>0$.
Thus $\frac{1}{n} \in A$.

Claim A is bounded above..

If $\frac{m}{n} \in A$ then $\frac{m}{n} < y$ so y is an upper bound of A.

Therefore $\alpha = \sup A \in \mathbb{R}$ completeness axiom of real numbers..

Let $\varepsilon > 0$ then $\alpha - \varepsilon$ is not an upper bound so there is $\frac{m}{n} \in A$ such that $\alpha - \varepsilon < \frac{m}{n}$

Let $\varepsilon = \frac{1}{n}$ then there is $\frac{m}{n} \in A$ st. $\alpha - \frac{1}{n} < \frac{m}{n}$.

Therefore $\alpha < \frac{m+1}{n}$

$\frac{m+1}{n} \leq x + \frac{1}{n} < y$

Recall $\frac{1}{n} < y - x$.

Thus $x + \frac{1}{n} < y$, and $\frac{m}{n} < y$, $\sup A < \frac{m+1}{n}$

Need to show $\frac{m}{n} > x$.

For contradiction suppose $\frac{m}{n} \leq x$. can't substitute here?

then $\frac{m+1}{n} \leq x + \frac{1}{n} < y$. this implies $\frac{m+1}{n} \in A$.

Since α is the least upper bound of A, then $\frac{m+1}{n} \leq \alpha$.

But since $\alpha < \frac{m+1}{n}$ this is a contradiction.

Thus $\frac{m}{n} > x$. so $\frac{m}{n} \in \mathbb{Q}$ and $\frac{m}{n} \in (x, y)$

so $\mathbb{Q} \cap (x, y) \neq \emptyset$.

Case $x<0$ and $y=0$
Case $x<y$ and $y<0$.

$\Big\}$ basically the same as cases already done by multiplying by -1 and applying previous argument

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<u>Cardinality</u>

$A \sim B$ means $\exists f: A \to B$ that's 1-to-1 and onto.

$A \sim B$ means $\exists f: A \to B$ that's a bijection.

<u>Definitions</u>:

- A is finite means there exists $n \in \mathbb{N}$ such that $A \sim \{1, 2, \dots, n\}$.
- A is infinite means not finite.
- A is countably infinite means $A \sim \mathbb{N}$
- A is countable means either finite or countably infinite.
- A is uncountable means that it's neither finite nor countable.

Easy results are called Propositions. Harder ones are Theorem.

Please finish reading §2.3 on Sup and Inf and then look at §2.4 Cardinality for next time.