Sup and Inf

① $\sup(A+B) = \sup A + \sup B$

where $A+B = \{a+b : a \in A \text{ and } b \in B\}$

② $\sup\{f(x)+g(x) : x \in X\} \le \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$

This is §2.3#6 and it is assigned.

Cardinality

Observe that if A has n elements, we can write A as $\{x_1, x_2, \ldots, x_n\}$. From the Pigeonhole Principle (Section 1.1, Exercise 8), which states that if there are m pigeons and n pigeonholes with $m > n$, then at least two pigeons must get in the same hole, it is clear that there cannot be a bijection from a finite set onto a proper subset of itself. From the paragraph preceding Definition 2.10, this is not the case with infinite sets.

means there is no bijection between a finite set and a strict subset of that finite set.

Proposition 2.8 Let A and B be sets.

1. If A is finite and $A \sim B$, then B is finite.
2. If B is infinite and $A \sim B$, then A is infinite.
3. If A is finite and $B \subset A$, then B is finite.
4. If B is infinite and $B \subset A$, then A is infinite.

Proof For part 1, first note that if A is empty, then so is B. Otherwise, $A \sim \{1, 2, \ldots, n\}$ for some n in $\mathbb{N}$. Since $\sim$ is transitive, $B \sim \{1, 2, \ldots, n\}$.

For part 4, note that it is the contrapositive of part 3. The rest of the proof is left as an exercise. ■

Proposition 2.9 Let A and B be sets.

1. If A is finite and there exists a function f from A onto B, then B is finite.
2. If A is infinite and there exists a one-to-one function from A into B, then B is infinite.

<u>Proposition</u> $\mathbb{N} \sim \{2,3,4,\dots\} = \mathbb{N} \setminus \{1\}$

That is $\mathbb{N}$ and $\mathbb{N} \setminus \{1\}$ have the same number of elements.

Bijection $\begin{cases} f: \mathbb{N} \to \mathbb{N} \setminus \{1\} \\ f(n) = n+1 \end{cases}$

<u>Proposition</u> If $B \subseteq A$ and A is countable then B is countable.

<u>Case</u>: If B is finite then it's countable.

Thus we can assume B is infinite. Since $B \subseteq A$ then A must be infinite. Since A is countable by hypothesis then $A \sim \mathbb{N}$. Therefore there is a function $f: \mathbb{N} \to A$ such that f is a bijection.

$$A = f(\mathbb{N}) = \{f(1), f(2), f(3), \dots\}$$

notation $x_i = f(i)$

$$A = \{x_1, x_2, x_3, \dots\}.$$

Since $\mathbb{N}$ is well ordered there is a smallest n_1 such that $x_{n_1} \in B$.

$$C = \{n : x_n \in B\} \subseteq \mathbb{N}$$

Let n_1 be the smallest such that $x_{n_1} \in B$

Let $n_2 > n_1$ be the smallest such that $x_{n_2} \in B$

$$C_1 = \{n > n_1 : x_n \in B\} \subseteq \mathbb{N}$$

Let $n_3 > n_2$ be the smallest such that $x_{n_3} \in B$

$\vdots$

Claim $B = \{x_{n_1}, x_{n_2}, x_{n_3} \dots \}$.

For contradiction, suppose $B \neq \{x_{n_1}, x_{n_2}, \dots\}$.

Clearly $B \supseteq \{x_{n_1}, x_{n_2}, \dots\}$ because they were chosen that way

Thus there must be a $b \in B$ such that $b \neq x_{n_i}$ for all $i \in \mathbb{N}$. Since $b \in B$ then $b \in A$ and so there is k such that $b = x_k$.

Case $k < n_1$ or there is i s.t. $n_{i-1} < k < n_i$

If $k < n_1$ then

Let n_1 be the smallest such that $x_{n_1} \in B$ would imply $n_1 \leq k$. Contradiction

If $n_{i-1} < k < n_i$ then

Let $n_i > n_{i-1}$ be the smallest such that $x_{n_i} \in B$ implies $n_i \leq k$. Contradiction.

Lemma: $\mathbb{N} \times \mathbb{N}$ is countable.

obviously not finite, so need to show $\mathbb{N} \times \mathbb{N}$ is countably infinite.

Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be given as

$$f(n,m) = 2^n 3^m.$$

Claim f is an injection (i.e. one-to-one)

Suppose $f(n,m) = f(r,s)$. Then

$$2^n 3^m = 2^r 3^s$$

Claim $n=r$ and $m=s$. Why?

Case $n>r$. Then $2^{n-r} = 3^{s-m}$.

since $n>r$ there are some 2's here

Thus 2 divides 3^{s-m}

so 2 divides either 1 or 3

which is a contradiction.

Case $n<r$. Then $2^{r-n} = 3^{m-s}$

Same argument gives a contradiction.

Case $n=r$. $2^{n-r} = 1 = 3^{s-m}$ so $s=m$.

<u>Recall</u> $f : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$

$$f(n,m) = 2^n 3^m$$

Define $B = f(\mathbb{N}\times\mathbb{N})$ so $f : \mathbb{N}\times\mathbb{N} \to B$ is a bijection

so $\mathbb{N}\times\mathbb{N} \sim B$

since $B \subseteq \mathbb{N}$ Thus B is countable

so $\mathbb{N}\times\mathbb{N}$ is countable.