HW1 due Friday, Feb 2

Turn in 1.1#1

Practice 1.1#7, 1.2#4, 1.2#10, 1.2#11, 1.3#1, 1.3#2

1. Prove that $\sqrt{6}$ is irrational.

For contradiction, suppose $\sqrt{6} = \frac{m}{n}$ where $m, n \in \mathbb{N}$ with no common divisors. It follows that

$$6n^2 = m^2$$

Since 6 is even, then m^2 is even. Moreover m must be even since the square of any odd number is again odd. Thus,

$$m = 2k \qquad \text{for some } k \in \mathbb{N}.$$

Substituting yields

$$6n^2 = (2k)^2 = 4k^2$$

and so

$$3n^2 = 2k^2.$$

Since 3 times an odd number is always odd, then it follows that n^2 is even and that n is even.

But then m even and n even contradicts the fact that that m and n have not common divisors.

It follows that $\sqrt{6}$ is irrational.

7. Let a be a real number. If $a^2 = a$, prove that either $a = 0$ or $a = 1$.

If $a \neq 0$ then one can divide both sides by a

$$\frac{a^2}{a} = \frac{a}{a}$$

to obtain that $a = 1$. Thus $a = 0$ or $a = 1$.

4. Let A and B be sets. The *symmetric difference* of A and B is $(A \cup B) \setminus (A \cap B)$. Show that $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$.

"$\subseteq$" Let $x \in (A \cup B) \setminus (A \cap B)$. Then

$$x \in A \cup B \quad \text{and} \quad x \notin A \cap B$$

Now $x \in A \cup B$ means $x \in A$ or $x \in B$

while $x \notin A \cap B$ means not $(x \in A$ and $x \in B)$.

By DeMorgans rule not $(x \in A$ and $x \in B)$ has the same truth values as $x \notin A$ or $x \notin B$. Thus

$$(x \in A \text{ or } x \in B) \text{ and } (x \notin A \text{ or } x \notin B)$$

Since $x \in A$ and $x \notin A$ is impossible as is $x \in B$ and $x \notin B$ it follows that either

$$(x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)$$

In other words

$$x \in A \setminus B \quad \text{or} \quad x \in B \setminus A.$$

Equivalently $x \in (A \setminus B) \cup (B \setminus A)$

"$\supseteq$" Suppose $x \in (A \setminus B) \cup (B \setminus A)$. Then

$$x \in A \setminus B \quad \text{or} \quad x \in B \setminus A.$$

Case $x \in A \setminus B$ then $x \in A$ and $x \notin B$.

Since $x \in A$ then clearly $x \in A \cup B$.

Furthermore since $x \notin B$ then $x \notin B \cap A$ because $B \cap A$ is a smaller set than B. Consequently

$$x \in A \cup B \quad \text{and} \quad x \notin B \cap A.$$

It follows that $x \in (A \cap B) \setminus (B \cap A)$.

To finish the proof we need to consider the other possibility

Case $x \in B \setminus A$ then $x \in B$ and $x \notin A$.

Since $x \in B$ then clearly $x \in A \cup B$.

Furthermore since $x \notin A$ then $x \notin B \cap A$ because $B \cap A$ is a smaller set than A. Consequently

$$x \in A \cup B \quad \text{and} \quad x \notin B \cap A.$$

It follows that $x \in (A \cap B) \setminus (B \cap A)$.

Since $x \in (A \cap B) \setminus (B \cap A)$ in both cases the proof is finished.

10. Let X be a set and let A_α be a set for each α in a nonempty index set I. Prove the distributive properties:

$$X \cap \left(\bigcup_{\alpha\in I} A_\alpha\right) = \bigcup_{\alpha\in I}(X \cap A_\alpha)$$

and

$$X \cup \left(\bigcap_{\alpha\in I} A_\alpha\right) = \bigcap_{\alpha\in I}(X \cup A_\alpha).$$

First we show $X \cap \left(\bigcup_{\alpha\in I} A_\alpha\right) = \bigcup_{\alpha\in I}(X \cap A_\alpha)$.

"$\subseteq$" Suppose $x \in X \cap \left(\bigcup_{\alpha\in I} A_\alpha\right)$. Then $x \in X$ and $x \in \bigcup_{\alpha\in I} A_\alpha$.

Now $x \in \bigcup_{\alpha\in I} A_\alpha$ means for some $\alpha_0 \in I$ that $x \in A_{\alpha_0}$.

Therefore $x \in X$ and $x \in A_{\alpha_0}$ implies $x \in X \cap A_{\alpha_0}$. Since $\alpha_0 \in I$ it immediately follows that $x \in \bigcup_{\alpha\in I}(X \cap A_{\alpha_0})$.

"$\supseteq$" Suppose $x \in \bigcup_{\alpha\in I}(X \cap A_\alpha)$. Then for some $\alpha_0 \in I$ it follows that $x \in X \cap A_{\alpha_0}$. Therefore $x \in X$ and $x \in A_{\alpha_0}$.

Since $x \in A_{\alpha_0}$ then $\alpha_0 \in I$ implies $x \in \bigcup_{\alpha\in I} A_\alpha$. Consequently

$$x \in X \text{ and } x \in \bigcup_{\alpha\in I} A_\alpha.$$

Therefore $x \in X \cap \bigcup_{\alpha\in I} A_\alpha$.

Next we show $X \cup \left(\bigcap_{\alpha \in I} A_\alpha \right) = \bigcap_{\alpha \in I} (X \cup A_\alpha)$.

"$\subseteq$" Suppose $x \in X \cup \left(\bigcap_{\alpha \in I} A_\alpha \right)$. Then

$$x \in X \quad \text{or} \quad x \in \bigcap_{\alpha \in I} A_\alpha.$$

Case $x \in X$ then $x \in X \cup A_\alpha$ for all $\alpha \in I$. Consequently

$x \in \bigcap_{\alpha \in I} (X \cup A_\alpha)$

Case $x \in \bigcap_{\alpha \in I} A_\alpha$ then $x \in A_\alpha$ for all $\alpha \in I$. Consequently

$x \in X \cup A_\alpha$ for all $\alpha \in I$. Therefore $x \in \bigcap_{\alpha \in I} (X \cup A_\alpha)$.

In both cases $x \in \bigcap_{\alpha \in I} (X \cup A_\alpha)$.

"$\supseteq$" Suppose $x \in \bigcap_{\alpha \in I} (X \cup A_\alpha)$. Then $x \in X \cup A_\alpha$ for all $\alpha \in I$.

Case $x \in X$ then $x \in X \cup \left(\bigcap_{\alpha \in I} A_\alpha \right)$.

Case $x \notin X$ then $x \in X \cup A_\alpha$ implies $x \in A_\alpha$ for all $\alpha \in I$

Therefore $x \in \bigcap_{\alpha \in I} A_\alpha$ and consequently $x \in X \cup \left(\bigcap_{\alpha \in I} A_\alpha \right)$

In both cases $x \in X \cup \left(\bigcap_{\alpha \in I} A_\alpha \right)$.

11. Let A, B, and C be sets. Prove that
$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

"$\subseteq$" Let $x \in A \times (B \cup C)$. Then $x = (x_1, x_2)$ where

$x_1 \in A$ and $x_2 \in B \cup C$.

Since $x_2 \in B \cup C$ then $x_2 \in B$ or $x_2 \in C$.

Case $x_2 \in B$. Then $x_1 \in A$ implies $x = (x_1, x_2) \in A \times B$

Consequently $x \in (A \times B) \cup (A \times C)$.

Case $x_2 \in C$. Then $x_1 \in A$ implies $x = (x_1, x_2) \in A \times C$.

Consequently $x \in (A \times B) \cup (A \times C)$

Since $x \in (A \times B) \cup (A \times C)$ in both cases then $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

"$\supseteq$" Let $x \in (A \times B) \cup (A \times C)$. Then $x \in A \times B$ or $x \in A \times C$.

Case $x \in A \times B$ then $x = (x_1, x_2)$ where $x_1 \in A$ and $x_2 \in B$.

Since $x_2 \in B$ then $x_2 \in B \cup C$. It follows that

$x = (x_1, x_2) \in A \times (B \cup C)$.

Case $x \in A \times C$ then $x = (x_1, x_2)$ where $x_1 \in A$ and $x_2 \in C$.

Since $x_2 \in C$ then $x_2 \in B \cup C$. It follows that

$x = (x_1, x_2) \in A \times (B \cup C)$.

Since $x \in A \times (B \cup C)$ in both cases then $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$

1. Let f be a function from X into Y. Let A and B be subsets of X. Prove that $f(A \cup B) = f(A) \cup f(B)$.

"$\subseteq$" Let $y \in f(A \cup B)$ then $y = f(x)$ where $x \in A \cup B$. Since $x \in A \cup B$ then $x \in A$ or $x \in B$.

Case $x \in A$. Then $y = f(x) \in f(A)$. Consequently $y \in f(A) \cup f(B)$.

Case $x \in B$. Then $y = f(x) \in f(B)$. Consequently $y \in f(A) \cup f(B)$.

In both cases $y \in f(A) \cup f(B)$. Therefore $f(A \cup B) \subseteq f(A) \cup f(B)$.

"$\supseteq$" Let $y \in f(A) \cup f(B)$. Then $y \in f(A)$ or $y \in f(B)$.

Case $y \in f(A)$. Then there is $a \in A$ such that $y = f(a)$.
since $a \in A \cup B$ it follows that $y = f(a) \in f(A \cup B)$.

Case $y \in f(B)$. Then there is $b \in B$ such that $y = f(b)$.
since $b \in A \cup B$ it follows that $y = f(b) \in f(A \cup B)$.

In both cases $y \in f(A \cup B)$. Therefore $f(A) \cup f(B) \subseteq f(A \cup B)$.

2. Define a function f from $\mathbb{R}$ into $\mathbb{R}$ by

$$f(x) = \begin{cases} -1 & \text{if} \quad x < 0 \\ x & \text{if} \quad x \geq 0. \end{cases}$$

Find each of the following.

(a) $f([-1, 1])$

(b) $f^{-1}([-1, 1])$

(c) $f^{-1}(\{-1\})$

(d) $f^{-1}(\{-2\})$

(e) $f^{-1}(f((-\infty, 0)))$

(f) $f(f^{-1}((-\infty, 0)))$

Graph of f:

(a) $f([-1,1]) = \{-1\} \cup [0,1]$

(b) $f^{-1}([-1,1]) = (-\infty,0) \cup [0,1] = (-\infty, 1]$

(c) $f^{-1}(\{-1\}) = (-\infty, 0)$

(d) $f^{-1}(\{-2\}) = \emptyset$

(e) $f^{-1}(f((-\infty,0))) = f^{-1}(\{-1\}) = (-\infty, 0)$

(f) $f(f^{-1}((-\infty,0))) = f((-\infty,0)) = \{-1\}$