HW3 due Friday, Feb 16
Turn in 2.2#3
Practice 2.1#7, 2.2#1abcd, 2.2#7, 2.2#8, 2.2#9

**7.** Prove parts 1 and 2 of Proposition 2.1.

**Proposition 2.1** Let $a$, $b$, and $c$ be in $\mathbb{R}$. Then

1. $|a| = 0$ if and only if $a = 0$;
2. $|-a| = |a|$;
3. $|ab| = |a||b|$;
4. if $c \geq 0$, then $|a| \leq c$ if and only if $-c \leq a \leq c$.

By definition $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 1 \end{cases}$

Proof of part 1.

"$\Rightarrow$" Suppose $|a|=0$. Claim that $a=0$. If not then either $a<0$ or $a>0$.

Case $a<0$. Then $|a| = -a > 0$ contradicting $|a|=0$.
Case $a>0$. Then $|a| = a > 0$ contradicting $|a|=0$.
Therefore $a=0$.

"$\Leftarrow$" Conversely suppose $a=0$ then $|a|=a=0$ by definition.

Proof of part 2.

Claim $|-a|=|a|$.

Case $a=0$. Then $|-a|=0$ and $|a|=0$ so $|-a|=|a|$.

Case $a<0$, Then $-a>0$ so

$$|-a| = \begin{cases} -a & \text{if } -a \geq 0 \\ -(-a) & \text{if } -a < 0 \end{cases} = -a$$

and

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases} = -a$$

Therefore $|-a| = |a|$.

Case $a \geq 0$. Then $-a \leq 0$ so

$$|-a| = \begin{cases} -a & \text{if } -a \geq 0 \\ -(-a) & \text{if } -a < 0 \end{cases} = -(-a) = a$$

and

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases} = a$$

So in all cases $|-a| = |a|$.

3. Let $A$ and $B$ be nonempty subsets of $\mathbb{R}$ with $A \subset B$. Show that

$\inf B \leq \inf A \leq \sup A \leq \sup B.$

Claim $\inf B \leq \inf A$. Since $A \subseteq B$ then any lower bound of $B$ is also a lower bound of $A$. Since $\inf B$ is a lower bound of $B$ then $\inf B$ is also a lower bound of $A$. Since $\inf A$ is the greatest of lower bounds of $A$ then $\inf B \leq \inf A$.

Claim $\inf A \leq \sup A$. Since $A \neq \emptyset$ then there exists $\alpha \in A$. Since $\inf A$ is a lower bound of $A$, then $\inf A \leq \alpha$. Also, since $\sup A$ is an upper bound, then $\alpha \leq \sup A$.

It follows that $\inf \alpha \leq \sup A$.

Note that if $A = \emptyset$ then $\inf \alpha = \infty$ and $\sup \alpha = -\infty$ in which case the inequality doesn't hold.

Claim $\sup A \leq \sup B$. Since $A \subseteq B$ then any upper bound of $B$ is also an upper bound of $A$. Since $\sup B$ is an upper bound of $B$ then $\sup B$ is also an upper bound of $A$. Since $\sup A$ is the least of upper bounds of $A$ then $\sup A \leq \sup B$.

1. Find the supremum and the infimum of each of the following sets.

(a) $\{x \in \mathbb{R} : 0 < x^2 < 2\}$

(b) $\{x \in \mathbb{R} : x^2 < 2\}$

(c) $\{x \in \mathbb{R} : 0 < x \text{ and } x^2 > 2\}$

(d) $\{x \in \mathbb{R} : x^2 > 2\}$

1a. $\sup\{x \in \mathbb{R} : 0 < x^2 < 2\} = \sup(-\sqrt{2}, 0) \cup (0, \sqrt{2}) = \sqrt{2}$

$\inf\{x \in \mathbb{R} : 0 < x^2 < 2\} = \inf(-\sqrt{2}, 0) \cup (0, \sqrt{2}) = -\sqrt{2}$

b. $\sup\{x : \mathbb{R} : x^2 < 2\} = \sup(-\sqrt{2}, \sqrt{2}) = \sqrt{2}$

$\inf\{x : \mathbb{R} : x^2 < 2\} = \inf(-\sqrt{2}, \sqrt{2}) = -\sqrt{2}$

c. $\sup\{x \in \mathbb{R} : 0 < x \text{ and } x^2 > 2\} = \sup(\sqrt{2}, \infty) = \infty$

$\inf\{x \in \mathbb{R} : 0 < x \text{ and } x^2 > 2\} = \inf(\sqrt{2}, \infty) = \sqrt{2}$

d. $\sup\{x \in \mathbb{R} : x^2 > 2\} = \sup(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty) = \infty$

$\inf\{x \in \mathbb{R} : x^2 > 2\} = \inf(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty) = -\infty$

**7.** Let $A$ and $B$ be subsets of $\mathbb{R}$. Show that

$$\sup(A \cup B) = \max\{\sup A, \sup B\}$$

and

$$\inf(A \cup B) = \min\{\inf A, \inf B\}.$$

If $A = \emptyset$ then the two statements reduce to

$$\sup B = \max\{-\infty, \sup B\} = \sup B$$

and

$$\inf B = \min\{\infty, \inf B\} = \inf B$$

which are obvious.

By symmetry the results hold true if $B = \emptyset$. Thus, we may assume that both $A \neq \emptyset$ and $B \neq \emptyset$.

"$\geq$" Since $A \subseteq A \cup B$ then $\sup A \leq \sup(A \cup B)$ similarly $B \subseteq A \cup B$ implies $\sup B \leq \sup(A \cup B)$. Thus

$$\sup(A \cup B) \geq \max\{\sup A, \sup B\}.$$

"$\leq$" Let $\gamma = \sup(A \cup B)$ and for contradiction suppose that $\gamma > \max\{\sup A, \sup B\}$. Since the inequality is strict there is $\varepsilon > 0$ such that $\gamma - \varepsilon > \max\{\sup A, \sup B\}$.

Thus $\gamma - \varepsilon > \sup A$ and $\gamma - \varepsilon > \sup B$.

(*) Since $\gamma - \varepsilon > \sup A$ then $\gamma - \varepsilon$ is an upper bound for $A$ and consequently $\alpha \leq \gamma - \varepsilon$ for all $\alpha \in A$.

(**) Since $\gamma-\varepsilon > \sup B$ then $\gamma-\varepsilon$ is an upper bound for $B$ and consequently $\beta \le \gamma-\varepsilon$ for all $\beta \in B$.

On the other hand $\gamma-\varepsilon < \sup(A\cup B)$. Therefore, $\gamma-\varepsilon$ is not an upper bound of $A\cup B$ and there exists $x \in A\cup B$ such that $\gamma-\varepsilon < x$.

Now $x \in A\cup B$ means either $x\in A$ or $x \in B$ or both.

Case $x \in A$ implies by (*) that $x \le \gamma-\varepsilon < x$ a contradiction

Case $x\in B$ implies by (**) that $x \le \gamma-\varepsilon < x$ which is also a contradiction.

Thus $\sup(A\cup B) \le \max\{\sup A, \sup B\}$.

It follows that $\sup(A\cup B) = \max\{\sup A, \sup B\}$.

To prove that $\inf(A\cup B) = \min\{\inf A, \inf B\}$ first note as in exercise 5 that $\inf(-S) = -\sup S$.

For completeness, I'll prove this result in a moment. Provided that $\inf(-S) = -\sup S$, then we can write

$$\inf(A\cup B) = -\sup(-A\cup B)$$

where $-A\cup B = \{-x : x\in A\cup B\}$.

since $\{-x : x \in A \cup B\} = \{x : x \in (-A) \cup (-B)\} = (-A) \cup (-B)$

where $-A = \{-\alpha : \alpha \in A\}$ and $-B = \{-\beta : \beta \in B\}$,

it follows from the previous part that

$$\sup((-A) \cup (-B)) = \max\{\sup(-A), \sup(-B)\}$$

and again by problem 5 that $\sup(-A) = -\inf(A)$ and $\sup(-B) = -\inf(B)$. Therefore

$$\sup((-A) \cup (-B)) = \max\{-\inf A, -\inf B\}.$$

Finally, the maximum of the negatives of two numbers is the negative of the minimum, so

$$\max\{-\inf A, -\inf B\} = -\min\{\inf A, \inf B\}.$$

Putting everything together yields

$$\inf(A \cup B) = -\left(-\min\{\inf A, \inf B\}\right) = \min\{\inf A, \inf B\}.$$

which was to be shown.

What's left is to show problem 5.

5. Let $S$ be a nonempty subset of $\mathbb{R}$ that is bounded above. Prove that $\inf(-S) = -\sup S$, where $-S = (-1)S = \{-s : s \in S\}$.

Let $\gamma = \inf(-S)$. It is enough to show $-\gamma = \sup S$.

Claim $-\gamma$ is an upper bound of $S$. By definition $\gamma$ is a lower bound for $-S$. Thus $\gamma \leq -s$ for all $s \in S$. It immediately follows that $-\gamma \geq s$ so $-\gamma$ is an upper bound.

Claim $-\gamma$ is the least upper bound of $S$. Let $\alpha$ be another upper bound. Then $s \leq \alpha$ for all $s \in S$ and so $-\alpha \leq -s$ for all $s \in S$. Thus $-\alpha$ is a lower bound of $-S$. Since $\gamma$ is the greatest lower bound $-\alpha \leq \gamma$. It follows that $-\gamma \leq \alpha$. Consequently $-\gamma$ is the least upper bound of $S$.

8. Show that a nonempty finite subset of $\mathbb{R}$ contains both a maximum and a minimum element. [*Hint:* Use induction.]

Let $S_n \subseteq \mathbb{R}$ be a subset of $\mathbb{R}$ such that $S_n \sim \{1, 2, \dots, n\}$. Thus $S_n$ is a finite subset of $\mathbb{R}$. Claim that $S_n$ has a maximum and minimum for all $n \in \mathbb{N}$. We prove this by induction.

Base case: If $n = 1$ then $S_1$ has only one element. Call that element $x_1$. Clearly $\max S_1 = x_1$ and $\min S_1 = x_1$ and so the maximum and minimum exist.

Induction step: Suppose any $S_n \subseteq \mathbb{R}$ such that $S_n \sim \{1, 2, \dots, n\}$ has a maximum and a minimum. Claim that any $S_{n+1} \subseteq \mathbb{R}$ such that $S_{n+1} \sim \{1, 2, \dots, n+1\}$ also has a maximum and minimum.

Let $S_{n+1} \subseteq \mathbb{R}$. There is a bijection $f: \{1, 2, \dots, n+1\} \to S_{n+1}$.
Let $A = f(\{1, 2, \dots, n\})$ and $B = f(\{n+1\})$.

Since $f$ is one-to-one it's a bijection onto it's image $A$. Consequently $A \sim \{1, 2, \dots, n\}$. From the induction hypothesis it follows that the maximum and minimum of $A$ exists.

Let $\alpha = \min A$ and $\beta = \max A$. By

**Proposition 2.3** Let $S$ be a subset of $\mathbb{R}^{\#}$.
1. If $S$ has a smallest element, then this smallest element is the infimum of $S$.
2. If $S$ has a greatest element, then this greatest element is the supremum of $S$.

it follows that $\alpha = \inf A$ and $\beta = \sup A$. Moreover, by Exercise 7 we have

$$\sup S_{n+1} = \sup A \cup B = \max\{\sup A, \sup B\} = \max\{\beta, f(n+1)\}.$$

Since $\beta \in S_{n+1}$ and $f(n+1) \in S_{n+1}$ it follows that

$$\gamma = \max\{\beta, f(n+1)\} \in S_{n+1}$$

Thus, $\gamma$ is the least upper bound of $S_{n+1}$ and $\gamma \in S_{n+1}$. By definition of the maximum, it follows that $\gamma = \max S_{n+1}$ and, in particular, that the maximum of $S_{n+1}$ exists.

The argument for minimum is similar. Namely

$$\inf S_{n+1} = \inf A \sqcup B = \min\{\inf A, \inf B\} = \min\{\alpha, f(n+1)\}.$$

Since $\alpha \in S_{n+1}$ and $f(n+1) \in S_{n+1}$ it follows that

$$\eta = \min\{\beta, f(n+1)\} \in S_{n+1}$$

Thus, $\eta$ is the greatest lower bound of $S_{n+1}$ and $\eta \in S_{n+1}$. By definition of the minimum, it follows that $\eta = \min S_{n+1}$ and, in particular, that the minimum of $S_{n+1}$ exists.

9. Let $A$ and $B$ be nonempty bounded subsets of $\mathbb{R}$, let $\alpha = \sup A$, and let $\beta = \sup B$. Let $C = \{ab : a \in A \text{ and } b \in B\}$. Show, by example, that $\alpha\beta \neq \sup C$ in general.

Let $A = \{-1, 2\}$ and $B = \{-3\}$. Thus

$$\alpha = \sup A = 2 \quad \text{and} \quad \beta = \sup B = -3$$

Now $C = \{ab : a \in A \text{ and } b \in B\} = \{3, -6\}$. Thus

$$\gamma = \sup C = 3.$$

However $\alpha\beta = 2(-3) = -6 \neq 3$.