HW4 due Friday, Feb 23

Turn in 2.3#6

Practice 2.3#3, 2.3#4, 2.4#2, 2.4#4, 2.4#10, 2.4#15

3. Find the sup and inf of

(a) $\{1 - 1/n : n \in \mathbb{N}\}$

(b) $\mathbb{Q}$

(c) $\{n - 1/n : n \in \mathbb{N}\}$

(d) $\{x \in \mathbb{Q} : x^2 < 2\}$.

(a) $\sup\{1-\frac{1}{n} : n \in \mathbb{N}\} = \sup\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\} = 1$

$\inf\{1-\frac{1}{n} : n \in \mathbb{N}\} = \inf\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\} = 0$

(b) $\sup \mathbb{Q} = \infty$ $\quad$ $\inf \mathbb{Q} = -\infty$

(c) $\sup\{n-\frac{1}{n} : n \in \mathbb{N}\} = \sup\{\frac{n^2-1}{n} : n \in \mathbb{N}\}$

$= \sup\{0, \frac{3}{2}, \frac{8}{3}, \frac{15}{4}, \dots\} = \infty$

$\inf\{n-\frac{1}{n} : n \in \mathbb{N}\} = 0$

(d) $\sup\{x \in \mathbb{Q} : x^2 < 2\} = \sup \mathbb{Q} \cap (-2, 2) = 2$ since $\mathbb{Q}$ is dense

$\inf\{x \in \mathbb{Q} : x^2 < 2\} = \inf \mathbb{Q} \cap (-2, 2) = -2$

4. Show that the irrational numbers are dense in $\mathbb{R}$. [*Hint:* Use the fact that $\sqrt{2}$ is irrational.]

Let $x, y \in \mathbb{R}$ with $x < y$ then we need to show there is $z \in \mathbb{R} \setminus \mathbb{Q}$ such that $z \in (x, y)$.

Consider the interval $\left(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}\right)$. Since $\frac{x}{\sqrt{2}} < \frac{y}{\sqrt{2}}$ it follows from Theorem 2.2 the density of $\mathbb{Q}$ in $\mathbb{R}$ that there is $\frac{p}{q} \in \mathbb{Q}$

such that $\frac{p}{q} \in \left(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}\right)$. Let $z = \frac{p}{q}\sqrt{2}$. Then $z \in (x, y)$.

What remains is to show that z is irrational.

Suppose, for contradiction, that z were rational. Then $z = \frac{m}{n}$ where $m, n \in \mathbb{Z}$. Consequently

$$\frac{m}{n} = z = \frac{p}{q}\sqrt{2}$$

would imply $\sqrt{2} = \frac{mq}{np} \in \mathbb{Q}$. However, it is known by Theorem 1.1 that $\sqrt{2}$ is irrational — a contradiction.

Therefore z is irrational and moreover the irrational numbers are dense in $\mathbb{R}$.

6. Let f and g be bounded functions from a nonempty set X into $\mathbb{R}$. Show that

$$\sup\{f(x) + g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}.$$

Show by examples that both equality and strict inequality can occur.

To prove that the least upper bound of $\{f(x) + g(x) : x \in X\}$ is less than $\sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$ it is enough to show that this is an upper bound.

Let $x \in X$. Then by definition $f(x) \leq \sup\{f(x) : x \in X\}$. Similarly we have that $g(x) \leq \sup\{g(x) : x \in X\}$.

Adding the above inequalities yields that

$$f(x)+g(x) \leq \sup\{f(x): x\in X\} + \sup\{g(x): x\in X\}$$

for all $x \in X$.

In otherwords $\sup\{f(x): x\in X\} + \sup\{g(x): x\in X\}$ is an upper bound for $\{f(x)+g(x): x\in X\}$.

As $\sup\{f(x)+g(x): x\in X\}$ is the least upper bound, it follows that

$$\sup\{f(x)+g(x): x\in X\} \leq \sup\{f(x): x\in X\} + \sup\{g(x): x\in X\}$$

which was to be proven.

2. Let f be a one-to-one function from A into B with B finite. Show that A is finite.

Let $C = f(A)$. By definition f is a bijection between C and A. Therefore $C \sim A$. On the other hand $C \subseteq B$ with B finite. From Proposition 2.8 part 3 it follows that C is finite. Since C and A have the same cardinality then A is also finite.

4. If X is an infinite set and x is in X, show that $X \sim X \setminus \{x\}$.

By theorem 2.3 since X is infinite then it contains a sequence $(x_n)_{n\in\mathbb{N}}$ of distinct points.

Let $A = \{x\} \cup \{x_1, x_2, \ldots\}$. Since the countable union of countable sets is countable then A is countable.

Let $B = A \setminus \{x\}$. Since $B \subseteq A$ then Proposition 2.11 part 3 implies that B is countable.

Claim that B is infinite. If not, then B would be finite. In particular $B \sim \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$ and there would be a bijection $f: \{1, 2, \ldots, n\} \to B$.

Define $g: \{1, 2, \ldots, n+1\} \to A$ as

$$g(k) = \begin{cases} f(k) & \text{for } k = 1, 2, \ldots, n \\ x & \text{for } k = n+1. \end{cases}$$

Since $A = B \cup \{x\} = f(\{1, \ldots, n\}) \cup \{x\}$ then g is onto. Since $x \notin B$ then g is one-to-one. It follows that g is a bijection and therefore $A \sim \{1, 2, \ldots, n+1\}$. This contradicts the fact that A is infinite. Therefore B is infinite.

Since B is infinite and countable there exists a bijection $h: \mathbb{N} \to B$. We are now able to define a bijection

$$g: X \to X \setminus \{x\}$$

as follows

$$g(w) = \begin{cases} h(1) & \text{if } w = x \\ h(k+1) & \text{if } w = h(k) \text{ for some } k \in \mathbb{N} \\ w & \text{otherwise} \end{cases}$$

Note g is well defined since h is one-to-one. What is left is to show that g is a bijection.

Claim g is onto. Let $y \in X \setminus \{x\}$.

If $y = h(1)$ then $g(x) = y$.

If $y = h(k)$ for $k > 1$ then $k-1 \in \mathbb{N}$ and so $g(h(k-1)) = y$.

otherwise $g(y) = y$.

In all cases there is an element in X that maps to y. Therefore g is onto.

Claim g is one-to-one. Suppose $g(a) = g(b)$. We need to show that $a = b$.

If $a = x$ then $g(a) = h(1)$. Consequently $g(b) = h(1)$. Now since h is one-to-one the only k such that $h(k) = g(b)$ is $k = 1$. Moreover the otherwise case in the definition of g could not happen because if it did then $g(b)=b$ would imply $b=h(1)$ in which case the evaluation of g would be by the first case. It follows that $b=x$, or that $a=b$.

If $a=h(k)$ for some $k\in\mathbb{N}$ then $g(a)=h(k+1)$ so $g(b)=h(k+1)$. Since h is one-to-one there is only one k such that $g(b)=h(k+1)$. Thus $b=h(k)=a$. Again note that the otherwise case can't happen.

If $a\neq x$ and $a\neq h(k)$ for all $k\in\mathbb{N}$ then $g(a)=a$. It follows that $g(b)=a$. Clearly $b\neq x$ for otherwise $g(b)=h(1)$ and $a\neq h(1)$. Similarly $b\neq h(k)$ for any k since in that case $g(b)=h(k+1)$ and $a\neq h(k+1)$. It follows that $g(b)=b$ by the otherwise case in the definition of g. Thus, $a=b$.

Since all possibilities for the evaluation of $g(a)$ lead to the consequence that $a=b$, then g is one-to-one.

Therefore $g: X\to X\setminus\{x\}$ is a bijection and $X\sim X\setminus\{x\}$.

10. Let A be an uncountable set and let B be a countable subset of A. Show that $A \setminus B$ is uncountable.

For contradiction suppose $A \setminus B$ were countable. Then

$$A = B \cup (A \setminus B)$$

would be a countable union of countable sets. By Theorem 2.4 that implies A is countable, which is a contradiction. Therefore $A \setminus B$ is uncountable.

15. Let A be the set of all real-valued functions on $[0, 1]$. Show that there does not exist a function from $[0, 1]$ onto A.

Let $\varphi : [0,1] \to A$. Claim that φ could not be onto. To do this we construct a function $f : [0,1] \to \mathbb{R}$ such that $\varphi(x) \neq f$ for every $x \in [0,1]$.

Since $\varphi(x) \in A$ then $\varphi(x) : [0,1] \to \mathbb{R}$ and it makes sense to evaluate $\varphi(x)$ at x denoted as $\varphi(x)(x) = (\varphi(x))(x)$

Now define $f(x) = \varphi(x)(x) + 1$

Note $f \neq \varphi(x)$ for every $x \in [0,1]$ since $f(x) \neq \varphi(x)(x)$.

Consequently φ is not onto.