HW5 due Friday, Mar 1

Turn in 3.1#5

Practice 3.1#2, 3.1#3, 3.1#9cd, 3.2#ef, 3.2#2ab, 3.2#6

2. Let $x_n = \begin{cases} 1/n & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$ Does $\lim_{n\to\infty} x_n$ exist?

Yes. The limit exists $\lim_{n\to\infty} x_n = 0$.

To see this let $\varepsilon > 0$ and choose n_0 such that $\frac{1}{n_0} < \varepsilon$. Then for $n \geq n_0$ we have since $0 < 1/n$ that

$$|x_n - 0| = \left\{ \begin{matrix} \frac{1}{n} \text{ if } n \text{ is odd} \\ 0 \text{ if } n \text{ is even} \end{matrix} \right\} \leq \frac{1}{n} \leq \frac{1}{n_0} < \varepsilon.$$

Therefore $\lim_{n\to\infty} x_n = 0$.

3. Let $x_n = \begin{cases} 1/n & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even.} \end{cases}$ Does $\lim_{n\to\infty} x_n$ exist?

No. The limit does not exist. Suppose it did. Then there would be $x \in \mathbb{R}$ such that $\lim_{n\to\infty} x_n = x$.

Let $\varepsilon = \frac{1}{3}$. By definition of limit it would follow there exists n_0 such that $n \geq n_0$ implies $|x_n - x| < 1/3$.

Let $n \geq \max\{n_0, 3\}$ be odd. Then $n+1$ is even so

$$|x_n - x| = \left|\frac{1}{n} - x\right| < 1/3 \quad \text{and} \quad |x_{n+1} - x| = |1 - x| < 1/3.$$

by definition

$$|x_n - x_{n+1}| = \left|\frac{1}{n} - 1\right| = 1 - \frac{1}{n}$$

and since $n \geq 3$ then $\frac{1}{n} \leq \frac{1}{3}$ or $-\frac{1}{n} \geq -\frac{1}{3}$. Thus,

$$|x_n - x_{n+1}| \geq 1 - \frac{1}{3} = \frac{2}{3}.$$

The triangle inequality now implies

$$\frac{2}{3} \leq |x_n - x_{n+1}| \leq |x_n - x| + |x_{n+1} - x| < \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

which is a contradiction. Therefore the limit does not exist.

5. Let $x_n \geq 0$ for each n in $\mathbb{N}$; let x be in $\mathbb{R}$ with $x_n \to x$. Note that $x \geq 0$. Show that $\sqrt{x_n} \to \sqrt{x}$. [*Hint:* Make two cases: $x = 0$ and $x > 0$. In the latter case, rationalize.]

Suppose $x_n \to x$ as $n \to \infty$. Claim that $x \geq 0$. For contradiction suppose $x < 0$. Then take $\varepsilon = |x|$ and n_0 large enough that $n \geq n_0$ implies $|x_n - x| < \varepsilon$. It follows that since $x_n \geq 0$ and $x < 0$ that

$$|x_n - x| = x_n - x < |x| = -x \quad \text{for } n \geq n_0$$

or that $x_n < 0$, which is a contradiction. Therefore we have that $x \geq 0$.

Case $x=0$. Then $x_n \to 0$ as $n \to \infty$.

Let $\varepsilon > 0$ it follows that $\varepsilon^2 > 0$ and so there is $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $|x_n - 0| < \varepsilon^2$ or equivalently $x_n < \varepsilon^2$.

Now estimate,

$$\left|\sqrt{x_n} - \sqrt{x}\right| = \left|\sqrt{x_n} - 0\right| = \sqrt{x_n} < \sqrt{\varepsilon^2} = \varepsilon.$$

Therefore $\lim\limits_{n \to \infty} \sqrt{x_n} = 0$

Case $x > 0$. Then $x_n \to x$ as $n \to \infty$.

Let $\varepsilon > 0$. choose $\varepsilon_1 = \varepsilon \sqrt{x}$.

It follows there is $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $|x_n - x| < \varepsilon_1$

Now estimate for $n \geq n_0$

$$\left|\sqrt{x_n} - \sqrt{x}\right| = \left|\left(\sqrt{x_n} - \sqrt{x}\right)\frac{\sqrt{x_n} + \sqrt{x}}{\sqrt{x_n} + \sqrt{x}}\right| = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}}$$

$$\leq \frac{|x_n - x|}{\sqrt{x}} \leq \frac{\varepsilon_1}{\sqrt{x}} = \varepsilon.$$

9. Establish the following limits.

(c) $\lim_{n\to\infty} n^{1/n} = 1$

By definition $n^{1/n} = \exp\left(\frac{1}{n}\log n\right)$. Following the remark on page 43 we assume the exponential function is continuous and use L'Hôpital's rule to evaluate the limit. Note both of these assumptions will be justified later.

$$\lim_{n\to\infty} \frac{\log n}{n} = \lim_{n\to\infty} \frac{\frac{d}{dn}(\log n)}{\frac{d}{dn} n} = \lim_{n\to\infty} \frac{1/n}{1} = 0$$

Therefore $\lim_{n\to\infty} \exp\left(\frac{1}{n}\log n\right) = \exp(0) = 1.$

(d) $\lim_{n\to\infty} (1+\frac{1}{n})^{2n} = e^2$

Again $\left(1+\frac{1}{n}\right)^{2n} = \exp\left(2n\log\left(1+\frac{1}{n}\right)\right)$ where exp is continuous. Computing as in Example 3.6 yields

$$\lim_{n\to\infty} n\log\left(1+\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\log\left(1+\frac{1}{n}\right)}{1/n} = \lim_{n\to\infty} \frac{\frac{d}{dn}\log(1+1/n)}{\frac{d}{dn}1/n}$$

$$= \lim \frac{\frac{1}{1+1/n}\left(0-\cancel{1/n^2}\right)}{-1/n^2} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1$$

Therefore $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{2n} = e^{2\cdot 1} = e^2.$

1. Find the limits of the following sequences.

(e) $(\sqrt{n} - \sqrt{n+1})_{n \in \mathbb{N}}$

By algebra

$$\sqrt{n} - \sqrt{n+1} = \left(\sqrt{n} - \sqrt{n+1}\right)\frac{\sqrt{n} + \sqrt{n+1}}{\sqrt{n} + \sqrt{n+1}} = \frac{n - (n+1)}{\sqrt{n} + \sqrt{n+1}} = \frac{-1}{\sqrt{n} + \sqrt{n+1}}$$

Therefore

$$\lim_{n \to \infty} \left(\sqrt{n} - \sqrt{n+1}\right) = \lim_{n \to \infty} \frac{-1}{\sqrt{n} + \sqrt{n+1}} = 0$$

(f) $\left(n - \sqrt{n^2 + n}\right)_{n \in \mathbb{N}}$

By algebra

$$n - \sqrt{n^2+n} = \left(n - \sqrt{n^2+n}\right)\frac{n + \sqrt{n^2+n}}{n + \sqrt{n^2+n}} = \frac{\cancel{n^2} - (\cancel{n^2} + n)}{n + \sqrt{n^2+n}} = \frac{-n}{n + \sqrt{n^2+n}} = \frac{-1}{1 + \sqrt{1 + 1/n}}$$

Therefore

$$\lim_{n \to \infty} \left(n - \sqrt{n^2+n}\right) = \lim_{n \to \infty} \frac{-1}{1 + \sqrt{1 + 1/n}} = \frac{-1}{\lim_{n \to \infty}\left(1 + \sqrt{1 + 1/n}\right)}$$

$$= \frac{-1}{1 + \lim_{n \to \infty} \sqrt{1 + 1/n}} = \frac{-1}{1 + \sqrt{1}} = \frac{-1}{2}$$

2. Give examples of two sequences that do not converge but whose
   (a) sum converges.
   (b) product converges.

(a) Let $x_n = n$ and $y_n = -n$. Then neither $(x_n)_{n\in\mathbb{N}}$ nor $(y_n)_{n\in\mathbb{N}}$ converge, but the sum

$$\lim_{n\to\infty}(x_n+y_n) = \lim_{n\to\infty}(n-n) = \lim_{n\to\infty} 0 = 0$$

converges.

(b) Let $x_n = (-1)^n$ and $y_n = (-1)^n$. Then neither $(x_n)_{n\in\mathbb{N}}$ nor $(y_n)_{n\in\mathbb{N}}$ converge, but the product

$$\lim_{n\to\infty} x_n y_n = \lim_{n\to\infty}(-1)^n(-1)^n = \lim_{n\to\infty}(-1)^{2n} = 1$$

converges.

6. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{R}$ (not necessarily convergent), and let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ with $y_n \to 0$. Show that $(x_n y_n)_{n\in\mathbb{N}}$ converges to 0.

Since $(x_n)_{n\in\mathbb{N}}$ is bounded there exists $B>0$ such that

$$|x_n| < B \quad \text{for all } n\in\mathbb{N}.$$

Let $\varepsilon>0$. Choose $\varepsilon_1 = \varepsilon/B$.

Since $y_n \to 0$ there is $n_0 \in \mathbb{N}$ such that

$|y_n - 0| < \varepsilon_1$ for all $n \geq n_0$.

Estimate

$$|x_n y_n - 0| = |x_n||y_n| \leq B|y_n| < B\varepsilon_1 = \varepsilon$$

This shows that $\lim_{n \to \infty} x_n y_n = 0$.