HW8 due Wednesday, May 1

Quiz 5 on Friday May 3 over homework

Practice 4.3#6, 4.4#6, 4.5#5, 4.6#10, 5.2#1abc, 5.2#3, 5.3#7

6. Let $f : D \to \mathbb{R}$ such that $\lim_{x \to c} f(x)$ is a real number. Show that there is a neighborhood U of c such that f is bounded on $U \cap D$.

Let $L \in \mathbb{R}$ such that $\lim_{x \to c} f(x) = L$. Then for $\varepsilon = 1$ there exists a neighborhood U of c such that $x \in U \cap D$ implies that $|f(x) - L| < \varepsilon$. Since

$$|f(x)| = |f(x) - L + L| \leq |f(x) - L| + |L| < \varepsilon + |L| = |L| + 1$$

then taking $B = |L| + 1$ yields f is bounded by B on $U \cap D$.

6. Show that a polynomial of even degree has either an absolute maximum or an absolute minimum.

Let $p(x)$ be a polynomial of even degree. Then

$$p(x) = \sum_{k=0}^{2n} a_k x^k \qquad \text{for some } n \in \mathbb{N} \text{ with } a_{2n} \neq 0.$$

Now

$$p(x) = x^{2n} \sum_{k=0}^{2n} a_k \frac{x^k}{x^{2n}} = x^{2n} \left(\sum_{k=0}^{2n-1} a_k \frac{1}{x^{2n-k}} + a_{2n} \right)$$

Consequently by the limit laws for extended real numbers

$$\lim_{x \to \infty} p(x) = \lim_{x \to \infty} x^{2n} \lim_{x \to} \left(\sum_{k=0}^{2n-1} a_k \frac{1}{x^{2n-k}} + a_{2n} \right)$$

$$= \infty \cdot a_{2n} = \begin{cases} \infty & \text{if } a_{2n} > 0 \\ -\infty & \text{if } a_{2n} < 0 \end{cases}$$

Similarly

$$\lim_{x\to-\infty} p(x) = \lim_{x\to\infty} x^{2n} \lim_{x\to} \left(\sum_{k=0}^{2n-1} a_k \frac{1}{x^{2n-k}} + a_{2n} \right)$$

$$= \infty \cdot a_{2n} = \begin{cases} \infty & \text{if } a_{2n} > 0 \\ -\infty & \text{if } a_{2n} < 0 \end{cases}$$

Now consider the two cases $a_{2n} > 0$ and $a_{2n} < 0$ separately.

Case $a_{2n} > 0$. Let $x_0 \in \mathbb{R}$ and define $y_0 = f(x_0)$

Since $\lim_{x\to\infty} P(x) = \infty$ there exists $b > c$ large enough

such that $x > b$ implies $P(x) > y_0$.

Since $\lim_{x\to-\infty} P(x) = \infty$ there exists $a < c$ small enough

such that $x < a$ implies $P(x) > y_0$.

Consider p restricted to the interval $[a,b]$. Since p is a polynomial it is continuous. By Theorem 4.2 it follows that p has an absolute minimum on $[a,b]$. Thus, there is $c \in [a,b]$ such that $p(c) \leq p(x)$ for all $x \in [a,b]$.

Claim $p(c)$ is an absolute minimum on all of $\mathbb{R}$.

If $x \in [a,b]$ we already know $p(c) \leq p(x)$. In particular since $x_0 \in [a,b]$ it follows that $p(c) \leq p(x_0) = y_0$. Now $x > b$ implies $p(c) \leq y_0 < p(x)$ and $x < a$ implies $p(c) \leq y_0 < p(x)$ shows $p(c)$ is the absolute minimum on $\mathbb{R}$.

Case $a_{2n} < 0$. Then $-p(x)$ is a polynomial of degree $2n$ that satisfies the conditions of the previous case.

It follows that $-p$ has an absolutute minimum at c such that $-p(c) \leq -p(x)$ for all $x \in \mathbb{R}$.

Equivalently $p(c) \geq p(x)$ for all $x \in \mathbb{R}$ so we see that $p(c)$ is an absolute maximum for p.

5. Show that $f(x) = 1/x^2$ is uniformly continuous on $[1, \infty)$, but not on $(0, 1]$.

Claim f is uniformly continuous on $[1, \infty)$.

Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{2}$. Then $x, y \in [1, \infty)$ and $|x-y| < \delta$ implies

$$\left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \left|\frac{y^2 - x^2}{x^2 y^2}\right| = |x-y| \frac{x+y}{x^2y^2}.$$

Now $x \geq 1$ implies $x^2y^2 \geq y^2$ and $y \geq 1$ implies $x^2y^2 \geq x^2$.
Therefore

$$x^2y^2 \geq \max(x^2, y^2) = \max(x,y)^2$$

Note also that $x+y \leq \max(x,y) + \max(x,y) = 2\max(x,y)$.
Consequently

$$\frac{x+y}{x^2y^2} \leq \frac{2\max(x,y)}{\max(x,y)^2} = \frac{2}{\max(x,y)} \leq 2$$

It follows that

$$\left|\frac{1}{x^2} - \frac{1}{y^2}\right| \leq 2|x-y| < 2\delta = 2\frac{\varepsilon}{2} = \varepsilon.$$

Claim f is not uniformly continuous on $(0, 1]$.

Suppose, for contradiction that f were uniformly continuous on $(0,1]$. Then by Theorem 4.5 it maps Cauchy sequences into Cauchy sequences.

Let $x_n = \frac{1}{n}$. Then $x_n \in (0,1]$ and $x_n \to 0$ as $n \to \infty$. Therefore $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. On the other hand $f(x_n) = \frac{1}{x_n^2} = n^2$ and so $(f(x_n))_{n\in\mathbb{N}}$ is unbounded.

This contradicts $(f(x_n))_{n\in\mathbb{N}}$ being Cauchy. It follows that f is not uniformly continuous.

10. Let f be a monotone function on I that satisfies the intermediate value property. Show that f is continuous on I.

If f were discontinuous at $c \in I$ that discontinuity would have to be of type 1.

Suppose f were monotone increasing.

Then either $f(c^-) < f(c)$ or $f(c^+) > f(c)$. Note in the case where c is an endpoint only one of the one-sided limits exist. Even so, it's always the case one of the above inequalities hold.

But that implies f doesn't satisfy the intermediate value property which is a contradiction.

If f is monotone decreasing consider $-f$ in place of f and apply the above argument to show $-f$ is continuous. Then $-f$ continuous implies f continuous which was to be shown.

1. Let $f : [0, 2] \to \mathbb{R}$ be continuous on $[0, 2]$ and differentiable on $(0, 2)$, with $f(0) = 0$ and $f(1) = f(2) = 1$.
   (a) Show that there is a c_1 in $(0, 1)$ with $f'(c_1) = 1$.
   (b) Show that there is a c_2 in $(1, 2)$ with $f'(c_2) = 0$.
   (c) Show that there is a c_3 in $(0, 2)$ with $f'(c_3) = \frac{1}{3}$.

(a) By the mean value theorem there exists $c_1 \in (0,1)$ such that $f(1) - f(0) = f'(c_1)(1-0)$

Substituting yields

$$1 - 0 = f'(c_1) \quad \text{so} \quad f'(c_1) = 1.$$

(b) By the mean value theorem there exists $c_2 \in (1,2)$ such that $f(2) - f(1) = f'(c_2)(2-1)$

Substituting yields

$$1 - 1 = f'(c_2) \quad \text{so} \quad f'(c_2) = 0.$$

(c) Since $f'(c_1) = 1$ and $f'(c_2) = 0$, by the intermediate value property for derivatives there is c_3 between c_1 and c_2 such that $f'(c_3) = 1/3$. Since $c_1 \in (0,1)$ and $c_2 \in (1,2)$ it follows that $c_3 \in (0,2)$.

3. Suppose that f is differentiable on $\mathbb{R}$ and that f has n distinct real roots. Show that f' has at least $n-1$ distinct real roots. Show by example that f' can have more real roots than f.

Suppose the distinct roots of f are $x_1, x_2, \ldots, x_n$ where $x_1 < x_2 < \cdots < x_n$. Consider the $n-1$ intervals

$$[x_1, x_2], \; [x_2, x_3], \; \ldots, \; [x_{n-1}, x_n].$$

Applying the mean value theorem on each interval implies there is $c_i \in [x_i, x_{i+1}]$ such that

$$f(x_{i+1}) - f(x_i) = f'(c_i)(x_{i+1} - x_i) \quad \text{for } i = 1, 2, \ldots, n-1$$

Since the x_i are roots, this is equivalent to

$$0 = f'(c_i)(x_{i+1} - x_i)$$

and so $f'(c_i) = 0$ for $i = 1, 2, \ldots, n-1$. In otherwords f' has $n-1$ roots.

To see f' might have more roots than f consider

$$f(x) = 13.$$

Then f has no roots but $f'(x) = 0$ for every $x \in \mathbb{R}$.

Alternatively $f(x) = x^2 + 1$ has no roots but $f'(x) = 2x$ has one root at $x = 0$.

7. If $x > 0$, show that

$$1 + x + \frac{x^2}{2} < e^x < 1 + x + \frac{x^2}{2}e^x.$$

By Taylor's theorem

$$e^x = 1 + x + \frac{e^c}{2}x^2 \quad \text{for some } c \in (0, x).$$

Since the exponential function is increasing

$$e^0 \leq e^c \leq e^x$$

It follows that

$$1 + x + \frac{e^0}{2}x^2 \leq 1 + x + \frac{e^c}{2}x^2 \leq 1 + x + \frac{e^x}{2}x^2.$$

or that

$$1 + x + \frac{1}{2}x^2 \leq e^x \leq 1 + x + \frac{e^x}{2}x^2.$$

which was to be shown.