HW2 due Friday, Feb 9

Turn in 1.4#10

Practice 1.3#8, 1.3#9, 1.4#2, 1.4#5, 2.1#2, 2.1#6

8. Let f be a function from X into Y. Let g be a function from Y into Z. Assume that $g \circ f$ is onto Z. Prove that g is onto Z. Give an example showing that f need not be onto Y.

Suppose $g \circ f : X \to Z$ is onto. We need to show g is onto.

Let $z \in Z$. Since $g \circ f$ is onto Z there is $x \in X$ such that $(g \circ f)(x) = z$. Define $y = f(x)$. Then

$$g(y) = g(f(x)) = (g \circ f)(x) = z$$

Therefore, given $z \in Z$ there is $y \in Y$ such that $g(y) = Z$. This shows g is onto.

A example where f is not onto is given by the maps $f : X \to Y$, $g : Y \to Z$ where

$$X = \{1\}, \quad Y = \{1, 2\}, \quad Z = \{1\}$$

and $f(x) = 1 \qquad g(y) = 1$

Note that f is not onto Y, however g is onto Z as is $g \circ f$.

9. Find a bijection from $(0, \infty)$ onto $(0, 1)$.

Let $f(x) = \frac{2}{\pi} \arctan x$. It actually takes a lot of calculus to show that f is a bijection onto $(0,1)$

An easier example is $f(x) = \frac{1}{x+1}$.

To see f is into $(0,1)$ let $x \in (0,\infty)$. Then $x+1 \in (1,\infty)$ and so $\frac{1}{x+1} \in (0,1)$.

To see $f(x)$ is onto let $y \in (0,1)$. Then $y = \frac{1}{x+1}$ implies $x = \frac{1}{y} - 1$. Moreover $y \in (0,1)$ implies $\frac{1}{y} \in (1,\infty)$

Therefore $x = \frac{1}{y} - 1 \in (0,\infty)$.

To see $f(x)$ is one-to-one suppose $f(a) = f(b)$. Then $\frac{1}{a+1} = \frac{1}{b+1}$. Thus $a+1 = b+1$ and finally $a = b$. It follows that f is one-to-one.

2. Prove that for each n in $\mathbb{N}$, $1^3 + 2^3 + \cdots + n^3 = [n(n+1)/2]^2$.

Let p_n be the statement "$1^3+2^3+\cdots+n^3 = (n(n+1)/2)^2$."
We use induction and show p_1 is true and p_n implies p_{n+1}.

Base case: p_1 true means "$1^3 = (1(1+1)/2)^2$". Since $1^3=1$ and $(1(1+1)/2)^2 = (2/2)^2 = 1$, this is true.

Induction step: Suppose p_n is true. That means
"$1^3+2^3+\cdots+n^3 = (n(n+1)/2)^2$"

We claim that p_{n+1} is true. By the induction hypothesis

$$1^3+2^3+\cdots+(n+1)^3 = 1^3+2^3+\cdots+n^3+(n+1)^3$$

$$= (n(n+1)/2)^2+(n+1)^3.$$

Now compute using algebra

$$(n(n+1)/2)^2+(n+1)^3 = n^2(n+1)^2/4 + n^3+3n^2+3n+1$$

$$= n^2(n^2+2n+1)/4+n^3+3n^2+3n+1$$

$$= \frac{n^4+2n^3+n^2}{4} + \frac{4n^3+12n^2+12n+4}{4}$$

$$= \frac{n^4+6n^3+13n^2+12n+4}{4}$$

on the other hand

$$((n+1)(n+2)/2)^2 = (n+1)^2(n+2)^2/4 = \frac{(n^2+2n+1)(n^2+4n+4)}{4}$$

problem §2.1#2 continues...

Now

$$\begin{array}{r} n^2+2n+1 \\ \times \quad n^2+4n+4 \\ \hline 4n^2+8n+4 \\ 4n^3+8n^2+4n \quad \\ + \quad n^4+2n^3+n^2 \qquad \\ \hline n^4+6n^3+13n^2+12n+4 \end{array}$$

shows

$$((n+1)(n+2)/2)^2 = \frac{n^4+6n^3+13n^2+12n+4}{4}$$

As this is the same as $(n(n+1)/2)^2+(n+1)^3$ it follows that

$$1^3+2^3+\dots+(n+1)^3 = ((n+1)(n+2)/2)^2$$

or that p_{n+1} holds.

By induction p_n holds for all $n \in \mathbb{N}$ which was to be shown.

5. Prove that for each n in $\mathbb{N}$, 4 divides $7^n - 3^n$.

Let p_n = " 4 divides $7^n - 3^n$ ". We show by induction that P_n holds for all $n \in \mathbb{N}$.

Base case: p_1 true means 4 divides $7^1 - 3^1 = 4$. This is clearly true since 4 divides 4 exactly once.

Induction step: Suppose p_n is true. We need to show p_{n+1}.
By hypothesis 4 divides $7^n - 3^n$. Thus there is $k \in \mathbb{Z}$ such that $7^n - 3^n = 4k$.

Since
$$x^{n+1} - y^{n+1} = x^n x - y^n y$$
$$= x^n x - x^n y + x^n y - y^n y = x^n(x-y) + (x^n - y^n)y$$
it follows using the induction hypothesis that
$$7^{n+1} - 3^{n+1} = 7^n(7-3) + (7^n - 3^n)3$$
$$= 7^n \cdot 4 + 4k \cdot 3 = 4(7^n + 3k).$$

Therefore 4 divides $7^{n+1} - 3^{n+1}$ and so p_{n+1} is true.

By induction p_n is true for all $n \in \mathbb{N}$ or in other words

4 divides $7^n - 3^n$ for all $n \in \mathbb{N}$.

10. Prove Bernoulli's inequality: If $x > -1$, then $(1+x)^n \geq 1+nx$ for each n in $\mathbb{N}$.

Suppose $x > -1$ and let $p_n =$ "$(1+x)^n \geq 1+nx$". We want to show by induction that p_n holds for all $n \in \mathbb{N}$.

Base Case: p_1 true means $(1+x)^1 \geq 1+1\cdot x$. Since the left and right sides are, in fact, equal then the inequality holds.

Induction step: Suppose p_n is true. Then $(1+x)^n \geq 1+nx$. Since $x > -1$ then $x+1 > 0$. Multiplying both sides of the inequality then yields

$$(1+x)(1+x)^n \geq (1+x)(1+nx)$$

or in other words that

$$(1+x)^{n+1} \geq 1+x+nx+nx^2 = 1+(n+1)x+nx^2$$

Thus $(1+x)^{n+1} \geq 1+(n+1)x$ and p_{n+1} holds.

By induction p_n holds for all $n \in \mathbb{N}$ or that

$$(1+x)^n \geq 1+nx \quad \text{for all } n \in \mathbb{N}.$$

Since we supposed $x > -1$ it follows that

$$\text{If } x > -1 \text{ then } (1+x)^n \geq 1+nx \text{ for each } n \in \mathbb{N}.$$

2. Show, by example, that if x and y are irrational, then $x+y$ and xy may be rational.

We know that $\sqrt{2}$ is irrational. Therefore $-\sqrt{2}$ is also irrational.
Let $x=\sqrt{2}$ and $y=-\sqrt{2}$. Then

$$x+y = \sqrt{2}+(-\sqrt{2}) = 0 \in \mathbb{Q}$$

and

$$xy = (\sqrt{2})(-\sqrt{2}) = -2 \in \mathbb{Q}.$$

6. Write $0.33474747\cdots$ as a fraction.

By definition

$$0.33474747\ldots = 0.33\overline{47}$$
$$= \frac{33}{100} + \frac{47}{10000}\left(1+\frac{1}{100}+\frac{1}{100^2}+\ldots\right)$$

To sum the geometric series let

$$S = 1+\frac{1}{100}+\frac{1}{100^2}+\ldots$$
$$\frac{1}{100}S = \frac{1}{100}+\frac{1}{100^2}+\frac{1}{100^3}+\ldots$$

$$\left(1-\frac{1}{100}\right)S = 1 \qquad \text{so} \qquad S=\frac{100}{99}$$

It follows that

$$0.33474747\ldots = \frac{33}{100}+\frac{47}{10000}\cdot\frac{100}{99} = \frac{1657}{4950}$$