Bolzano Weierstrass theorem

a bounded sequence in $\mathbb{R}$ has a convergent subsequence

Vector version

a bounded sequence in $\mathbb{R}^n$ has a convergent subsequence

Proof from 310

(a) Every sequence has a monotone subsequence.

(b) Every bounded monotone subsequence converges..

Let $x_k \in \mathbb{R}^n$ be bounded.

write $x_k = \begin{bmatrix} x_{k,1} \\ x_{k,2} \\ \vdots \\ x_{k,n} \end{bmatrix} = (x_{k,1}, x_{k,2}, \dots, x_{k,n})$

Note $x_{k,1} \in \mathbb{R}$ and so there is a monotone subsequence $x_{k_j,1}$

The Monotone Subsequence Theorem

The following proof is taken from Bartle and Sherbert because the authors could not improve on their elegant argument.

Theorem 3.9 (Monotone Subsequence Theorem) Every sequence in $\mathbb{R}$ has a monotone subsequence.

Proof Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. For the purpose of this proof, we call the mth term x_m a *peak* if $x_m \geq x_n$ for all $n \geq m$. That is, x_m is a peak if x_m is never exceeded by any term that follows it.

Case 1 $(x_n)_{n\in\mathbb{N}}$ has infinitely many peaks. We pick off the peaks in order. Let m_1 be the smallest positive integer such that x_{m_1} is a peak. Let m_2 be the smallest positive integer larger than m_1 such that x_{m_2} is a peak. Continuing, we obtain the subsequence $(x_{m_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$. Since each x_{m_k} is a peak, we have $x_{m_1} \geq x_{m_2} \geq \cdots$ and hence $(x_{m_k})_{k\in\mathbb{N}}$ is monotone decreasing.

Case 2 $(x_n)_{n\in\mathbb{N}}$ has only a finite number of peaks. Let the peaks be (in order) $x_{m_1}, x_{m_2}, x_{m_3}, \ldots, x_{m_r}$. We go out beyond the last peak. Let $n_1 = m_r + 1$ (if the number of peaks is 0, let $n_1 = 1$.) Since x_{n_1} is not a peak, there is an $n_2 > n_1$ such that $x_{n_1} < x_{n_2}$. Since x_{n_2} is not a peak, there is an $n_3 > n_2$ such that $x_{n_2} < x_{n_3}$. Continuing, we obtain a strictly increasing subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$. ■

Attempt for a Proof of B.W. theorem in $\mathbb{R}^n$

Let $x_k \in \mathbb{R}^n$ be bounded.

write $x_k = \begin{bmatrix} x_{k,1} \\ x_{k,2} \\ \vdots \\ x_{k,n} \end{bmatrix} = (x_{k,1}, x_{k,2}, \ldots, x_{k,n})$

Note $x_{k,1} \in \mathbb{R}$ and so there is a monotone subsequence $x_{k_j,1}$

Now consider $x_{k_j,2} \in \mathbb{R}$.

so there is a monotone subsubsequence $x_{k_{j_i},2}$

Now consider $x_{k_{j_i},3} \in \mathbb{R}$.

so there is a monotone subsubsubsequence $x_{k_{j_{i_l}},3}$

⋮

After $(\text{sub})^n$-sequences each component of the vectors in that $(\text{sub})^n$-sequence are monotone (and bounded since the vectors were bounded to start with).

Claim the (sub)^n-sequence of vectors converges...

By reindexing let x_{m_j} denote the (sub)^n-sequence.

we know $x_{m_j,1} \to l_1$ as $j \to \infty$ since that component is monotone

$x_{m_j,2} \to l_2$ as $j \to \infty$

$\vdots$

$x_{m_j,n} \to l_n$ as $j \to \infty$

Let $\varepsilon > 0$. Then there is K_1 so large that

$$|l_1 - x_{m_j,1}| < \varepsilon/\sqrt{n} \quad \text{for} \quad j \geq K_1$$

$$|l_2 - x_{m_j,2}| < \varepsilon/\sqrt{n} \quad \text{for} \quad j \geq K_2$$

$$\vdots$$

$$|l_n - x_{m_j,n}| < \varepsilon/\sqrt{n} \quad \text{for} \quad j \geq K_n$$

Now let $l = (l_1, l_2, \ldots, l_n)$ then $j \geq \max(K_1, K_2, \ldots, K_n)$ implies

$$|l - x_{m_j}| = \left\| \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{bmatrix} - \begin{bmatrix} x_{m_j,1} \\ x_{m_j,2} \\ \vdots \\ x_{m_j,n} \end{bmatrix} \right\| = \sqrt{(l_1 - x_{m_j,1})^2 + (l_2 - x_{m_j,2})^2 + \cdots + (l_n - x_{m_j,n})^2}$$

$$< \sqrt{\underbrace{\left(\frac{\varepsilon}{\sqrt{n}}\right)^2 + \left(\frac{\varepsilon}{\sqrt{n}}\right)^2 + \cdots + \left(\frac{\varepsilon}{\sqrt{n}}\right)^2}_{n\text{-term}}} = \sqrt{\varepsilon^2} = \varepsilon,$$