1.25 Theorem. *The connected subsets of $\mathbb{R}$ are precisely the intervals (open, half-open, or closed; bounded or unbounded).*

"$\Rightarrow$" Easy

"$\Leftarrow$" If S is an interval then it's connected.

Case $S = [a,b]$. ← Closed & bounded

Suppose for contradiction S were disconnected.

Thus, there are sets $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$ such that

$$[a,b] = S_1 \cup S_2 \quad \text{and} \quad \overline{S_1} \cap S_2 = \emptyset \quad \text{and} \quad S_1 \cap \overline{S_2} = \emptyset.$$

Relabeling, if necessary, suppose $b \in S_2$.

Let $c = \sup\{S_1\}$. Thus $c \in \overline{S_1} \subseteq \overline{S} = \overline{[a,b]} = [a,b]$

Since $c \in \overline{S_1}$ then $\overline{S_1} \cap S_2 = \emptyset$ implies $c \notin S_2$

Since $c \notin S_2$ then $c \neq b$ and so $c < b$.

Since c is an upper bound on S_1 then all points bigger than c which are in S are actually in S_2.

this interval exists since $c < b$.

Thus $(c,b] \subseteq S_2$ so $[c,b] \subseteq \overline{S_2}$

This implies $c \in \overline{S_2}$. But then $\overline{S_2} \cap S_1 = \emptyset$ means $c \notin S_1$. Since $c \in S = S_1 \cup S_2$ then it must be that $c \in S_2$. Which contradicts $c \notin S_2$

Case S is a non-compact interval.

Suppose for contradiction that there are sets $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$ such that

$S = S_1 \cup S_2$ and $\overline{S_1} \cap S_2 = \emptyset$ and $S_1 \cap \overline{S_2} = \emptyset$.

Since $S_1 \neq \emptyset$ there is $a \in S_1$

$S_2 \neq \emptyset$ there is $b \in S_2$

Relabeling if necessary we assume $a < b$.

Since S is an interval, this means $[a,b] \subseteq S$.

Define $T_1 = S_1 \cap [a,b]$ and $T_2 = S_2 \cap [a,b]$.

Claim T_1, T_2 is a disconnection for $[a,b]$.

$$T_1 \cup T_2 = (S_1 \cap [a,b]) \cup (S_2 \cap [a,b]) = (S_1 \cup S_2) \cap [a,b] = S \cap [a,b] = [a,b]$$

since $[a,b] \subseteq S$.

$a \in T_1$ so $T_1 \neq \emptyset$

$b \in T_2$ so $T_2 \neq 0$

$$\overline{T_1} \cap T_2 = \overline{S_1 \cap [a,b]} \cap S_2 \cap [a,b] \subseteq \overline{S_1} \cap [a,b] \cap S_2 \cap [a,b] = \emptyset$$

Question $\overline{A \cap B} \ \overset{?}{=} \ \overline{A} \cap \overline{B}$

$\subseteq$ Try to prove this from the definitions and or sequence theorem about closures.

$\supseteq$

Example: $A = \mathbb{Q}$ $B = \mathbb{R} \setminus \mathbb{Q}$ $\overline{A \cap B} = \overline{\emptyset} = \emptyset$

rational irrationals. $\overline{A} \cap \overline{B} = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$

Similarly $T_1 \cap \overline{T_2} = \emptyset$

Therefore T_1 and T_2 is a disconnection of $[a,b]$.
But from the first case we know $[a,b]$ is connected.

Theorem: If $S \subseteq \mathbb{R}^m$ is connected and
$f: S \to \mathbb{R}^n$ is continuous.

Then $f(S)$ is connected.

Proof: By contradiction suppose $f(S)$ is disconnected.
Then there is a disconnection of $f(S)$

$U_1 \neq \emptyset$, $U_2 \neq \emptyset$, $f(S) = U_1 \cup U_2$, $\overline{U_1} \cap U_2 = \emptyset$ and $U_1 \cap \overline{U_2} = \emptyset$.

Let $S_1 = f^{-1}(U_1) = \{x \in S : f(x) \in U_1\}$ Closure is a limiting process...

$S_2 = f^{-1}(U_2) = \{x \in S : f(x) \in U_2\}$

details we use somewhere that f is continuous.

Therefore S_1 and S_2 is a disconnection of S.
This contradicts S being connected.

Intermediate value theorem: Let $S \subseteq \mathbb{R}^n$ and $f: S \to \mathbb{R}$ be continuous. If $V \subseteq S$ is connected then
$a, b \in V$ with $f(a) < t < f(b)$ or $f(b) < t < f(a)$
implies there is $c \in V$ such that $f(c) = t$.

Proof $f(V)$ is connected by the previous theorem.
Since $f(V) \subseteq \mathbb{R}$ then it's an interval by the theorem before that. Thus there is c such that $f(c) = t$.