Path connected.

There is another notion of connectedness that is important in many situations. A set $S \subset \mathbb{R}^n$ is called **arcwise connected** (or **pathwise connected**) if any two points in $S$ can be joined by a continuous curve in $S$, that is, if for any $\mathbf{a}, \mathbf{b}$ in $S$ there is a continuous map $\mathbf{f} : [0, 1] \to \mathbb{R}^n$ such that $\mathbf{f}(0) = \mathbf{a}$, $\mathbf{f}(1) = \mathbf{b}$, and $\mathbf{f}(t) \in S$ for all $t \in [0, 1]$.

path


$$h(t) = \begin{cases} f(2t) & 0 \le t \le 1/2 \\ g(2t-1) & 1/2 \le t \le 1 \end{cases}$$

**1.28 Theorem.** *If $S \subset \mathbb{R}^n$ is arcwise connected, then $S$ is connected.*

between every two points in $S$ is a path.

ie. $S$ is not disconnected

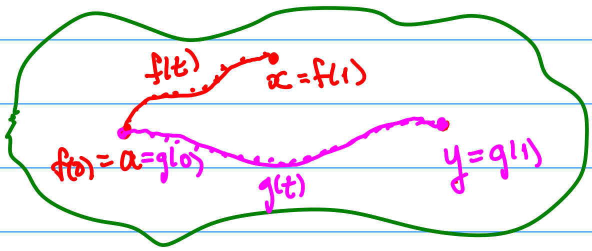
Proof: For contradiction suppose $S$ is disconnected...

*Proof.* We shall assume that $S$ is disconnected and show that it is not arcwise connected. Accordingly, suppose $(S_1, S_2)$ is a disconnection of $S$. Pick $\mathbf{a} \in S_1$ and $\mathbf{b} \in S_2$; we claim that there is no continuous $\mathbf{g} : [0, 1] \to S$ such that $\mathbf{g}(0) = \mathbf{a}$ and $\mathbf{g}(1) = \mathbf{b}$. If there were, the set $V = \mathbf{g}([0, 1])$ would be connected by Theorems 1.25 and 1.26. But this cannot be so: $V$ is the union of $V \cap S_1$ and $V \cap S_2$; these sets are nonempty since $\mathbf{a} \in V \cap S_1$ and $\mathbf{b} \in V \cap S_2$, and neither of them intersects the closure of the other. Hence $S$ is not arcwise connected. $\square$

**1.30 Theorem.** *If $S \subset \mathbb{R}^n$ is open and connected, then $S$ is arcwise connected.*

Proof: Let $a \in S$ and $S_1 = \{x : \text{there is a path between } a \text{ and } x\}$.

Note $S_1$ is arcwise connected $\quad x, y \in S_1$




$h : [0,1]$ continuous...
$h(0) = x \quad h(1) = y$
$h(t) \in S_1$ for all $t \in [0,1]$

write out the definition of $h$ explicitly using cases...

For contradiction,
suppose $S$ were not arcwise connected. Then $S_1 \neq S$ and in particular $S_2 = S \setminus S_1 \neq \emptyset$.

Claim that $S_1, S_2$ is a disconnection for $S$ (contradicting that $S$ is connected).

Already know $S_1 \neq \emptyset$, $S_2 \neq \emptyset$, $S = S_1 \cup S_2$.

Need to show $\bar{S}_1 \cap S_2 = \emptyset$ and $S_1 \cap \bar{S}_2 = \emptyset$.

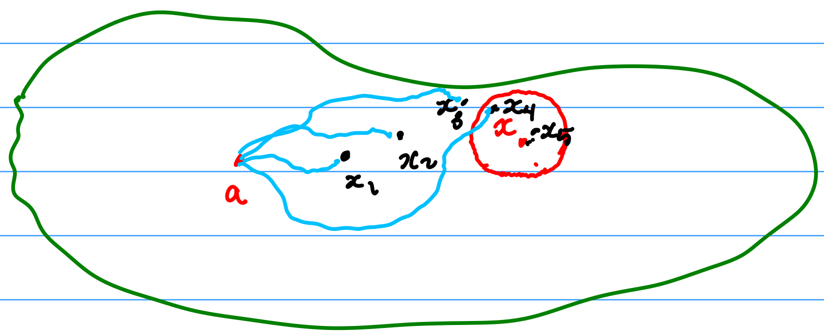
Claim $S_1 \cap \bar{S}_2 = \emptyset$. Let $x \in S_1$. Since $x \in S$ and $S$ is open there is a ball of radius $r$ about $x$ contained in $S$. Thus $B(r, x) \subseteq S$.

There is a path from $a$ to $x$ since $x \in S_1$.


Draw a straight line between $x$ and $y$. This a path between $x$ and any point $y \in B(r, x)$.

Thus every $y \in B(r,x)$ is in $S_1$ or $B(r,x) \subseteq S_1$. Thus $\overline{S_2}$ could not contain any points of $S_1$ since those points are all interior points of $S_1$. Thus $S_1 \cap \overline{S_2} = \emptyset$.

Claim $\overline{S_1} \cap S_2 = \emptyset$. Let $x \in \overline{S_1}$ also in $S$.




Let $x_j \in S_1$ with $x_j \to x$. Since they converge then for some $K$ large enough $x_j \in B(r,x)$ for $j \geq K$.

There is a path from $a$ to $x_K$ since $x_K \in S_1$.
There is a path from $x_K$ to $x$ since $x_K \in B(r,x)$
straight line.

Therefore $x \in S_1$. (terminology $S_1$ is a clopen set. both closed and open relative to $S$. )

Then $S_2 = S \setminus S_1$ implies $x \notin S_2$. So $\overline{S_1} \cap S_2 = \emptyset$.

Again this a contradiction because $S$ is connected.

# Section 1.8 Uniform Continuity

the condition for $\mathbf{f}$ to be continuous on $S$ is that

$$(1.31)\quad \forall\epsilon>0\ \forall\mathbf{x}\in S\ \exists\delta>0:\ \forall\mathbf{y}\in S\quad |\mathbf{x}-\mathbf{y}|<\delta \implies |\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})|<\epsilon,$$

whereas the condition for $\mathbf{f}$ to be uniformly continuous on $S$ is that

$$(1.32)\qquad \forall\epsilon>0\ \exists\delta>0:\ \forall\mathbf{x},\mathbf{y}\in S\quad |\mathbf{x}-\mathbf{y}|<\delta \implies |\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})|<\epsilon.$$

**1.33 Theorem.** *Suppose $S\subset\mathbb{R}^n$ and $\mathbf{f}: S\to\mathbb{R}^m$ is continuous at every point of $S$. If $S$ is compact, then $\mathbf{f}$ is uniformly continuous on $S$.*

recall: Closed intervals $[a,b]$ are examples of compact sets.

For contradiction suppose not, then

not $\left(\forall\epsilon>0\ \exists\delta>0:\ \forall\mathbf{x},\mathbf{y}\in S\quad |\mathbf{x}-\mathbf{y}|<\delta \implies |\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})|<\epsilon.\right)$

$$\exists\varepsilon>0\ \forall\delta>0\ \exists x,y\in S\ |x-y|<\delta \Rightarrow |f(x)-f(y)|\geq\varepsilon$$

Let $\varepsilon$ be such an $\varepsilon$ fixed....

Let $\delta_k=\frac{1}{k}$ then $x_k, y_k\in S$ $\quad |x_k-y_k|<\frac{1}{k}$

and $|f(x_k)-f(y_k)|\geq\varepsilon$

By B.W. there is a conv. subseq. $x_{k_j}\to a$

Since $S$ is closed $a\in S$.

Since $|x_{k_j}-y_{k_j}|<\frac{1}{k_j}$ then $y_{k_j}\to a$ also.

Now $f(x_{k_j})-f(y_{k_j})\to f(a)-f(a)=0$ since

$f$ is continuous at $a$. Contradicting $|f(x_{k_j})-f(y_{k_j})|\geq\varepsilon$.