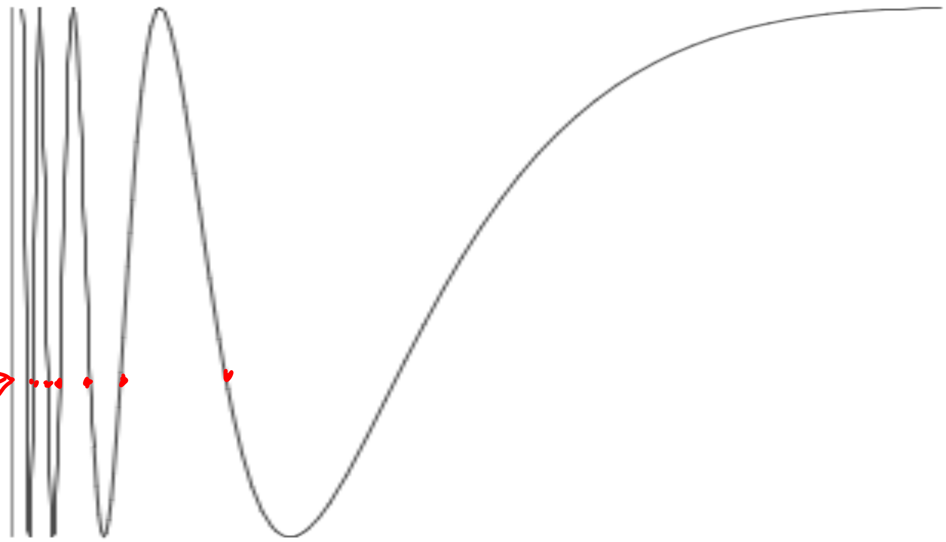
converges to here so to the vertical line.



$S = \{(x, y) : 0 < x \leq 2 \text{ and } y = \sin(\pi/x)\} \cup \{(0, y) : y \in [-1, 1]\},$

- as you walk along the sine curve towards the left it is infinitely long and so you can never reach a point on the vertical line. So not path connected.

- Since the sine curve gets closer and closer to the vertical line it's closure contains the vertical line. So there is no disconnection and the set S is therefore connected.

So, without any additional hypothesis connected does not imply path connected.

Chapter 2

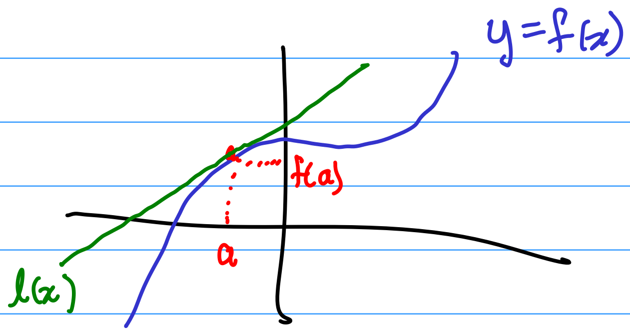
DIFFERENTIAL CALCULUS

Let $f : \mathbb{R} \to \mathbb{R}$. Then f is differentiable at a if the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. (set $h = x - a$)

The original idea had something to do with tangent lines



line passing through $(a, f(a))$

$l(x) = f(a) + m(x - a)$.

by definition

$m = f'(a)$

the slope of the tangent.

$x - a = h$

Thus

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = m$$

or

$$\lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} - m \right) = 0$$

$$\lim_{h \to 0} \left(\frac{f(a+h) - f(a) - mh}{h} \right) = 0$$

original curve

tangent line

$$E(h) = \underbrace{f(a+h)}_{\text{original curve}} - \underbrace{\left(f(a) + mh\right)}_{\text{tangent line}}$$

error in the tangent line approximation.

just a linear approximation...

Thus, $\displaystyle\lim_{h\to 0} \frac{E(h)}{h} = 0$

Says something about how fast the error in the tangent line approximation goes to zero..

Just means this limit is zero

Notation: write $E(h) = o(h)$

little-oh of h.

In higher dimensions we let $l(h)$ be a linear approximation involving matrix and or vector multiplication...

Let $f : \mathbb{R}^n \to \mathbb{R}$

Recall a linear function from $\mathbb{R}^n \to \mathbb{R}^m$ is given by matrix mult. by $A \in \mathbb{R}^{m\times n}$

In this case $m = 1$ so we can use a column vector... any linear function from $\mathbb{R}^n \to \mathbb{R}$ is given by $c^T x = c \cdot x$ for some $c \in \mathbb{R}^n$.

Thus we write the linear approximation at the point $(a, f(a))$ as

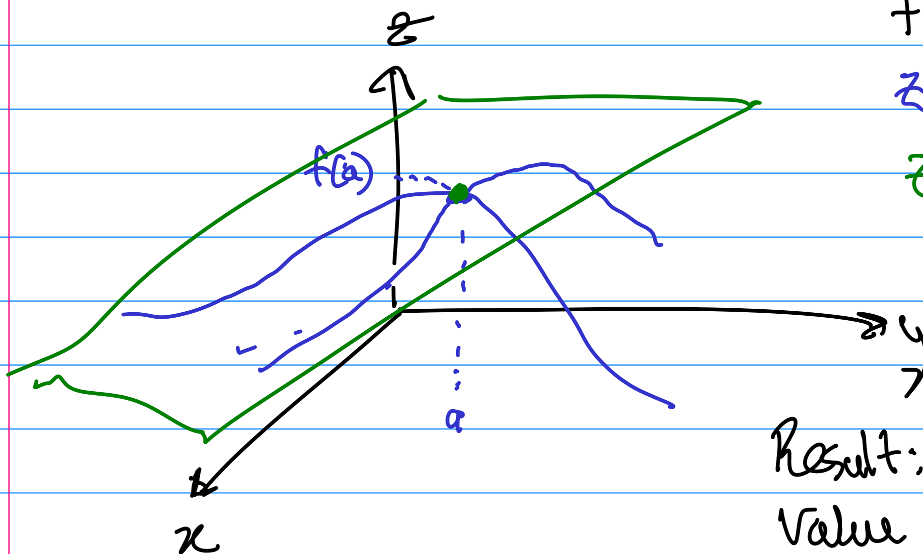
$$l(x) = f(a) + c\cdot(x-a).$$

We look for a good approximation so $E(h) = o(\|h\|)$.

$f: \mathbb{R}^2 \to \mathbb{R}$

$z = f(x,y)$

$z = l(x,y)$



Result: there might be a value of c such that

$$E(h) = o(\|h\|)$$

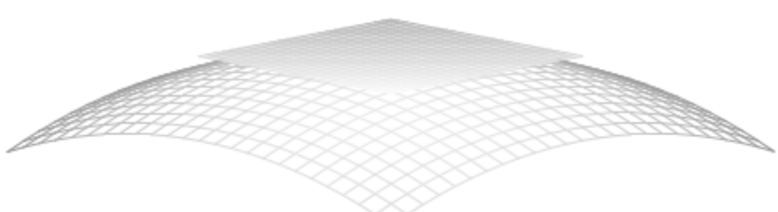
If there is, that value is unique and called the derivative of f at a.

Notation $\nabla f(a) = c$.

Warning If the function is differentiable then

$$\nabla f = (\partial_1 f, \partial_2 f, \dots, \partial_n f)$$

But there are cases where the partial derivatives exist but f is not differentiable ...



2.17 Theorem. *If f is differentiable at* $\mathbf{a}$, *then the partial derivatives* $\partial_j f(\mathbf{a})$ *all exist, and they are the components of the vector* $\nabla f(\mathbf{a}) = c$.

Suppose c exists and chosen so $E(h) = o(|h|)$.

$$\lim_{\vec{h} \to 0} \frac{f(a+\vec{h}) - f(a) - c \cdot \vec{h}}{|\vec{h}|} = 0$$

Let $\vec{h} = e_j h$ ← scalar ... . The above limit implies

$$\lim_{h \to 0} \frac{f(a+e_j h) - f(a) - c \cdot e_j h}{\pm |e_j h|} = 0$$

Note $|e_j h| = |h|$

since the limit is zero it doesn't matter what the sign is.

Thus

$$\lim_{h \to 0} \frac{f(a+e_j h) - f(a) - c_j h}{h} = 0$$

Limit laws

$$\partial_j f(a) = \lim_{h \to 0} \frac{f(a+e_j h) - f(a)}{h} = c_j$$

(definition of partial derivative)

2.18 Theorem. *If f is differentiable at $\mathbf{a}$, then f is continuous at $\mathbf{a}$.*

Proof. Multiplying (2.15) through by $|\mathbf{h}|$, we see that $f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \cdot \mathbf{h} \to 0$ as $\mathbf{h} \to \mathbf{0}$. Since $\nabla f(\mathbf{a}) \cdot \mathbf{h}$ clearly vanishes as $\mathbf{h}$ does, we have $f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) \to 0$ as $\mathbf{h} \to \mathbf{0}$, which says precisely that f is continuous at $\mathbf{a}$. □

Next time prove ...

2.19 Theorem. *Let f be a function defined on an open set in $\mathbb{R}^n$ that contains the point $\mathbf{a}$. Suppose that the partial derivatives $\partial_j f$ all exist on some neighborhood of $\mathbf{a}$ and that they are continuous at $\mathbf{a}$. Then f is differentiable at $\mathbf{a}$.*