Let $P_{a,k}(h) = \sum_{j=0}^{k} \frac{f^{(j)}(a)}{j!} h^j,$ and $R_{a,k}(h) = f(a+h) - P_{a,k}(h)$:

2.55 Theorem (Taylor's Theorem with Integral Remainder, I). *Suppose that f is of class C^{k+1} $(k \geq 0)$ on an interval $I \subset \mathbb{R}$, and $a \in I$. Then the remainder $R_{a,k}$ defined by* (2.53)–(2.54) *is given by*

$$R_{a,k}(h) = \frac{h^{k+1}}{k!} \int_0^1 (1-t)^k f^{(k+1)}(a+th)\, dt. \tag{2.56}$$

2.58 Theorem (Taylor's Theorem with Integral Remainder, II). *Suppose that f is of class C^k $(k \geq 1)$ on an interval $I \subset \mathbb{R}$, and $a \in I$. Then the remainder $R_{a,k}$ defined by* (2.53)–(2.54) *is given by*

$$R_{a,k}(h) = \frac{h^k}{(k-1)!} \int_0^1 (1-t)^{k-1} \left[f^{(k)}(a+th) - f^{(k)}(a) \right] dt. \tag{2.59}$$

2.63 Theorem (Taylor's Theorem with Lagrange's Remainder). *Suppose f is $k+1$ times differentiable on an interval $I \subset \mathbb{R}$, and $a \in I$. For each $h \in \mathbb{R}$ such that $a+h \in I$ there is a point c between 0 and h such that*

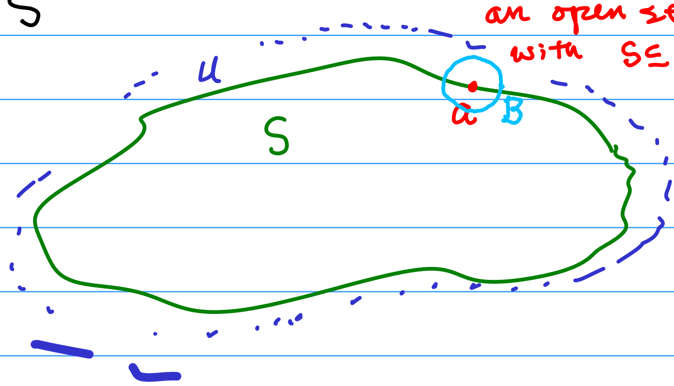
$$R_{a,k}(h) = f^{(k+1)}(a+c) \frac{h^{k+1}}{(k+1)!}. \tag{2.64}$$

again
from Th

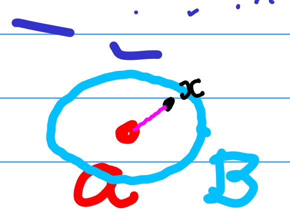
Let $f : \mathbb{R}^n \to \mathbb{R}$ where $f \in C^{k \text{ or } k+1}(S)$ and $S \subseteq \mathbb{R}^n$

given $a \in S$

means there is an open set U with $S \subseteq U$ and $f \in C^{k \text{ or } k+1}(U)$



$B \subseteq U$



$h = x - a$

$\ell(t) = a + th$

$\ell(0) = a$ $\ell(1) = x$

$$g(t) = f(\ell(t)) = f(a + th)$$

Apply Taylor's Theorem to g.

$$\underset{h=1}{g(1)} = \underset{a}{g(0)} + 1\,g'(0) + \frac{1^2}{2} g''(0) + \cdots + \frac{1^k}{k!} g^{(k)}(0) + R_k$$

$$= \sum_{j=0}^{k} \frac{g^{(j)}(0)}{j!} + \begin{cases} \frac{1}{k!} \int_0^1 (1-t)^k g^{(k+1)}(t)\,dt \\ \frac{1}{(k-1)!} \int_0^1 (1-t)^{k-1} \left(g^{(k)}(t) - g^{(k)}(0) \right) dt \\ \frac{1}{(k+1)!} g^{(k+1)}(c) \quad \text{for some } c \text{ between } 0 \text{ and } 1. \end{cases}$$

rewrite the above in terms of f.

$$g(t) = f(\ell(t)) = f(a + th)$$

$$g'(t) = \nabla f(a+th) \cdot h = h \cdot \underbrace{\nabla f}_{\text{linear approx. given in definition}}(a+th) = \underbrace{(h \cdot \nabla)}_{\text{partial derivatives}} f(a+th)$$

$$g'(t) = (h_1\partial_1 + h_2\partial_2 + \cdots + h_n\partial_n) f(a+th)$$

$$g^{(j)}(t) = \underbrace{(h_1\partial_1 + h_2\partial_2 + \cdots + h_n\partial_n)^j}_{\text{simplify using multinomial theorem} \ldots} f(a+th)$$

$x_1 = h_1\partial_1$
$x_2 = h_2\partial_2$

$$(x_1 + x_2 + \cdots + x_n)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{x}^\alpha.$$

$$g^{(j)}(t) = \sum_{|\alpha|=j} \frac{j!}{\alpha!} (h_1\partial_1, h_2\partial_2, \cdots, h_n\partial_n)^\alpha f(a+th)$$

$$g^{(j)}(t) = \sum_{|\alpha|=j} \frac{j!}{\alpha!} (h_1\partial_1)^{\alpha_1}(h_2\partial_2)^{\alpha_2} \cdots (h_n\partial_n)^{\alpha_n} f(a+th)$$

$$= \sum_{|\alpha|=j} \frac{j!}{\alpha!} \left(h_1^{\alpha_1} h_2^{\alpha_2} \cdots h_n^{\alpha_n}\right)\left(\partial_1^{\alpha_1}\partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}\right) f(a+th)$$

$$= \sum_{|\alpha|=j} \frac{j!}{\alpha!} h^\alpha \partial^\alpha f(a+th)$$

Thus

$$f(x) = f(a+h) = \sum_{j=0}^{k} \frac{1}{j!} \sum_{|\alpha|=j} \frac{j!}{\alpha!} h^\alpha \partial^\alpha f(a) \quad + R$$

$$= \sum_{|\alpha|\le k} \frac{h^\alpha}{\alpha!} \partial^\alpha f(a) \quad + \begin{cases} \frac{1}{k!}\int_0^1 (1-t)^k \sum_{|\alpha|=k+1} \frac{(k+1)!}{\alpha!} h^\alpha \partial^\alpha f(a+th)\, dt \\ \frac{1}{(k-1)!}\int_0^1 (1-t)^{k-1} \sum_{|\alpha|=k} \frac{k!}{\alpha!} h^\alpha\left(\partial^\alpha f(a+th) - f(a)\right) dt \\ \frac{1}{(k+1)!} \sum_{|\alpha|=k+1} \frac{(k+1)!}{\alpha!} h^\alpha \partial^\alpha f(a+ch) \quad \text{for some } c\in(0,1) \end{cases}$$

$$= \sum_{|\alpha|\le k} \frac{h^\alpha}{\alpha!} \partial^\alpha f(a) \quad + \begin{cases} (k+1) \sum_{|\alpha|=k+1} \frac{h^\alpha}{\alpha!} \int_0^1 (1-t)^k \partial^\alpha f(a+th)\, dt \quad ✓ \\ k \sum_{|\alpha|=k} \frac{h^\alpha}{\alpha!} \int_0^1 (1-t)^{k-1} \left(\partial^\alpha f(a+th) - f(a)\right) dt \quad ✓ \\ \sum_{|\alpha|=k+1} \frac{h^\alpha}{\alpha!} \partial^\alpha f(a+ch) \quad \text{for some } c\in(0,1) \quad ✓ \end{cases}$$

2.75 Corollary. *If f is of class C^k on S, then $R_{\mathbf{a},k}(\mathbf{h})/|\mathbf{h}|^k \to 0$ as $\mathbf{h} \to \mathbf{0}$. If f is of class C^{k+1} on S and $|\partial^\alpha f(\mathbf{x})| \leq M$ for $\mathbf{x} \in S$ and $|\alpha| = k+1$, then*

$$|R_{\mathbf{a},k}(\mathbf{h})| \leq \frac{M}{(k+1)!}\|\mathbf{h}\|^{k+1},$$

where

$$\|\mathbf{h}\| = |h_1| + |h_2| + \cdots + |h_n|.$$

Proof: Let $f \in C^k(S)$

$$R(h) = k \sum_{|\alpha|=k} \frac{h^\alpha}{\alpha!} \int_0^1 (1-t)^{k-1} \left(\partial^\alpha f(a+th) - f(a)\right) dt$$

Since $f \in C^k(S)$ then the partials $\partial^\alpha f$ are continuous

Given $\varepsilon > 0$ and $|\alpha| = k$ there is $\delta_\alpha > 0$ such that

$$|x-y| < \delta_\alpha \text{ implies } |\partial^\alpha f(x) - \partial^\alpha f(y)| < \varepsilon$$

Let $\delta = \min\{\delta_\alpha : |\alpha| = k\}$. If $|h| < \delta$ then

$$|R(h)| \leq k \sum_{|\alpha|=k} \frac{|h^\alpha|}{\alpha!} \int_0^1 (1-t)^{k-1} \left|\partial^\alpha f(a+th) - f(a)\right| dt$$

$$< k \sum_{|\alpha|=k} \frac{|h^\alpha|}{\alpha!} \int_0^1 (1-t)^{k-1} \varepsilon \, dt = \sum_{|\alpha|=k} \frac{|h^\alpha|}{\alpha!} \varepsilon$$

$$= \sum_{|\alpha|=k} \frac{|h_1^{\alpha_1} h_2^{\alpha_2} \cdots h_n^{\alpha_n}|}{\alpha!} \varepsilon = \sum_{|\alpha|=k} \frac{|h_1|^{\alpha_1} |h_2|^{\alpha_2} \cdots |h_n|^{\alpha_n}}{\alpha!} \varepsilon$$

$$= \left(|h_1| + |h_2| + \cdots + |h_n|\right)^k \varepsilon \text{ by the multinomial theorem.}$$

need to compare to $\sqrt{|h_1|^2 + |h_2|^2 + \cdots |h_n|^2}$

(note $\varepsilon \to 0$ as $h \to 0$)