Recall

Given $\varepsilon > 0$ and $|\alpha| = k$ there is $\delta_\alpha > 0$ such that

$$|x-y| < \delta_\alpha \text{ implies } |\partial^\alpha f(x) - \partial^\alpha f(y)| < \varepsilon$$

Let $\delta = \min\{\delta_\alpha : |\alpha| = k\}$. If $|h| < \delta$ then

$$|R(h)| \leq k \sum_{|\alpha|=k} \frac{|h^\alpha|}{\alpha!} \int_0^1 (1-t)^{k-1} |\partial^\alpha f(a+th) - f(a)| \, dt$$

$$< k \sum_{|\alpha|=k} \frac{|h^\alpha|}{\alpha!} \int_0^1 (1-t)^{k-1} \varepsilon \, dt = \sum_{|\alpha|=k} \frac{|h^\alpha|}{\alpha!} \varepsilon$$

$$= \sum_{|\alpha|=k} \frac{|h_1^{\alpha_1} h_2^{\alpha_2} \cdots h_n^{\alpha_n}|}{\alpha!} \varepsilon = \sum_{|\alpha|=k} \frac{|h_1|^{\alpha_1} |h_2|^{\alpha_2} \cdots |h_n|^{\alpha_n}}{\alpha!} \varepsilon$$

$$= (|h_1| + |h_2| + \cdots + |h_n|)^k \varepsilon$$ by the multinomial theorem.

need to compare to $\sqrt{|h_1|^2 + |h_2|^2 + \cdots |h_n|^2}$

(note $\varepsilon \to 0$ as $h \to 0$)

Thus

$$|R(h)| \leq (|h_1| + \cdots + |h_n|)^k \varepsilon \quad \text{for } |h| < \delta.$$

$$|h| = \sqrt{|h_1|^2 + \cdots + |h_n|^2}$$

$$|h_1| \leq \sqrt{|h_1|^2 + \cdots + |h_n|^2}$$

$$|h_2| \leq \sqrt{|h_1|^2 + \cdots + |h_n|^2}$$

$$\vdots$$

$$|h_n| \leq \sqrt{|h_1|^2 + \cdots + |h_n|^2}$$

So

$$|h_1| + |h_2| + \cdots + |h_n| \le n\sqrt{|h_1|^2 + \cdots + |h_n|^2} = n|h|$$

Then

$$|R(h)| \le (|h_1| + \cdots + |h_n|)^k \varepsilon \le (n|h|)^k \varepsilon = n^k |h|^k \varepsilon$$

so $\dfrac{|R(h)|}{|h|^k} \le n^k \varepsilon$ for all $|h| < \delta$

Since $\varepsilon > 0$ was arbitrary, it follows that

$$\frac{|R(h)|}{|h|^k} \to 0 \quad \text{as} \quad h \to 0.$$

2.75 Corollary. *If f is of class C^k on S, then $R_{\mathbf{a},k}(\mathbf{h})/|\mathbf{h}|^k \to 0$ as $\mathbf{h} \to \mathbf{0}$. If f is of class C^{k+1} on S and $|\partial^\alpha f(\mathbf{x})| \le M$ for $\mathbf{x} \in S$ and $|\alpha| = k+1$, then*

$$|R_{\mathbf{a},k}(\mathbf{h})| \le \frac{M}{(k+1)!}\|\mathbf{h}\|^{k+1},$$

where

$$\|\mathbf{h}\| = |h_1| + |h_2| + \cdots + |h_n|.$$

$$f(a+h) = \sum_{|\alpha| \le k} \frac{h^\alpha}{\alpha!} \partial^\alpha f(a) + R_k(h)$$

Let $k = 2$

$$f(a+h) = \boxed{f(a) +} \sum_{|\alpha|=1} \frac{h^\alpha}{\alpha!} \partial^\alpha f(a) + \sum_{|\alpha|=2} \frac{h^\alpha}{\alpha!} \partial^\alpha f(a) + R_k(h)$$

comes from $\sum_{|\alpha|=0}$

$$\alpha \in (\mathbb{N} \cup \{0\})^n \qquad |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$$

order of α multiindex

If $|\alpha| = 1$ then $\quad \alpha = (1, 0, 0, \ldots, 0)$

or $\quad \alpha = (0, 1, 0, \ldots, 0)$

$\vdots$

$\alpha = (0, 0, \ldots, 1)$

Therefore

$$\sum_{|\alpha|=1} \frac{h^\alpha}{\alpha!} \partial^\alpha f(a) = \sum_{j=1}^{n} h_i \partial_i f(a) = h \cdot \nabla f(a)$$

If $|\alpha| = 2$ then $\quad \alpha = (2, 0, \ldots, 0)$

$\alpha = (0, 2, \ldots, 0)$

$\vdots$

$\alpha = (0, 0, \ldots, 2)$

and $\quad \alpha = (1, 1, \ldots, 0)$

$\alpha = (1, 0, 1, \ldots, 0)$

$\vdots$

$$\sum_{|\alpha|=1} \frac{h^\alpha}{\alpha!} \partial^\alpha f(a) = \sum_{j=1}^{n} \frac{h_j^2}{2} \partial_j^2 f(a) + \sum_{i<j} h_i h_j \partial_i \partial_j f(a)$$

$$= \sum_{j=1}^{n} \frac{h_j^2}{2} \partial_j^2 f(a) + \frac{\sum_{i<j} h_i h_j \partial_i \partial_j f(a) + \sum_{i<j} h_i h_j \partial_i \partial_j f(a)}{2}$$

$$= \sum_{j=1}^{n} \frac{h_j^2}{2} \partial_j^2 f(a) + \frac{\sum_{i<j} h_i h_j \partial_i \partial_j f(a) + \sum_{i>j} h_i h_j \partial_i \partial_j f(a)}{2}$$

$$\sum_{|\alpha|=1} \frac{h^\alpha}{\alpha!}\partial^\alpha f(a) = \frac{1}{2}\sum_{j} h_i \underbrace{\sum_{j} \partial_i \partial_j f(a)\, h_j}_{\text{matrix vector product}}$$

Let
$$H = \begin{bmatrix} \partial_1\partial_1 f(a) & \partial_1\partial_2 f(a) & \cdots & \partial_1\partial_n f(a) \\ \vdots & & & \\ \partial_n\partial_1 f(a) & \cdots & \cdots & \partial_n\partial_n f(a) \end{bmatrix}$$

$$\sum_{|\alpha|=1} \frac{h^\alpha}{\alpha!}\partial^\alpha f(a) = \frac{1}{2}\, h \cdot Hh$$

note the book used k instead of h here...

Therefore

$$f(a+h) = \boxed{f(a) +} \sum_{|\alpha|=1} \frac{h^\alpha}{\alpha!}\partial^\alpha f(a) + \sum_{|\alpha|=2} \frac{h^\alpha}{\alpha!}\partial^\alpha f(a) + R_2(h)$$

$$= f(a) + h\cdot \nabla f(a) + \frac{1}{2}\, h\cdot Hh + R_2(h)$$

2.78 Proposition. *If f has a local maximum or minimum at* $\mathbf{a}$ *and f is differentiable at* $\mathbf{a}$*, then* $\nabla f(\mathbf{a}) = \mathbf{0}$.

Let $g(t) = f(a+tu)$ where $u \in \mathbb{R}^n$ is a unit vector.

Then $g: \mathbb{R} \to \mathbb{R}$ so we can apply one-dimensional results.

Moreover $g(0) = f(a)$ so if $f(a)$ is a max of f then $g(0)$ is a max of g.

Theorem # from your Math 310 textbook...

over the weekend review

If $g: \mathbb{R} \to \mathbb{R}$ is diff at t_0 and $g(t_0)$ is a local extremum of g, then $g'(t_0) = 0$.

Now, by chain rule

directional derivative ...

$$g'(0) = \nabla f(g(0)) \cdot u = \partial_u f(a) = 0$$

since u is a unit vector

Now taking $u = e_1, e_2, \ldots, e_n$ we obtain

$$\partial_1 f(a) = 0, \quad \partial_2 f(a) = 0, \ldots, \partial_n f(a) = 0$$

and since f is differentiable, then

$$\nabla f(a) = (\partial_1 f(a), \partial_2 f(a), \ldots, \partial_n f(a)) = 0,$$

2nd derivative test... in multiple dimensions...

Hessian Matrix ..

matrix vector pro c

$$\text{Let} \quad H = \begin{bmatrix} \partial_1\partial_1 f(a) & \partial_1\partial_2 f(a) & \cdots & \partial_1\partial_n f(a) \\ \vdots & & & \\ \partial_n\partial_1 f(a) & \cdots & \cdots & \partial_n\partial_n f(a) \end{bmatrix}$$

If can switch order of partial derivatives then $H = H^T$.

From linear algebra..

<u>Spectral Theorem</u>: If $H \in \mathbb{R}^{n \times n}$ and $H = H^T$, then there is an orthonormal basis of $\mathbb{R}^n$ made of eigenvectors of H.

Thus

$$H\xi_i = \lambda_i \xi_i \quad \text{and} \quad \xi_i \cdot \xi_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

ξ_i eigenvector with eigenvalue λ_i

orthonormal basis

$$\text{Let} \quad Q = \left[\xi_1 \middle| \xi_2 \middle| \cdots \middle| \xi_n\right] \in \mathbb{R}^{n \times n} \qquad QQ^T = I$$

Since Q is square then $Q^{-1} = Q^T$.

pivot in every row means a pivot in every column...

$$HQ = \left[\lambda_1\xi_1 \,\middle|\, \lambda_2\xi_2 \,\middle|\, \cdots \,\middle|\, \lambda_n\xi_n\right] = QD \quad \text{where } D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}$$

Therefore $HQ = QD$ means $H = QDQ^{-1} = QDQ^T$.

Suppose $f \in C^2(S)$ and a in the interior and $\nabla f(a) = 0$.
Then

$$f(a+h) = f(a) + h \cdot \nabla f(a) + \tfrac{1}{2} h \cdot Hh + R_2(h)$$

$$f(a+h) = f(a) + \tfrac{1}{2} h \cdot QDQ^T h + R_2(h)$$

$$= f(a) + \tfrac{1}{2} Q^T h \cdot D Q^T h + R_2(h)$$

$c = Q^T h$

$$= f(a) + \underbrace{\tfrac{1}{2} c \cdot Dc}_{\text{this needs to be non negative.}} + R_2(h)$$

positive

$$c \cdot Dc = \sum_{i=1}^{n} \lambda_i c_i^2$$

all these positive

what means for $f(a)$ to be a minimum?

$\ell = \min \lambda_i$

$$c \cdot Dc \geq \ell \sum_{i=1}^{n} c_i^2 = \ell\, c \cdot c = \ell\, Q^T h \cdot Q^T h = \ell\, h \cdot QQ^T h$$

$$= \ell |h|^2$$

Now $R_2(h) \leq n^2 |h|^2 \varepsilon$ so choosing ε small enough that $n^2 \varepsilon < l/4$ implies

$$f(a+h) \geq f(a) + \frac{l}{2}|h|^2 - n^2|h|^2\varepsilon \geq f(a) + \underbrace{\frac{l}{4}}_{\text{positive.}}|h|^2$$

So $f(a)$ is a relative minimum...