3.1 Theorem (The Implicit Function Theorem for a Single Equation). *Let $F(\mathbf{x}, y)$ be a function of class C^1 on some neighborhood of a point $(\mathbf{a}, b) \in \mathbb{R}^{n+1}$. Suppose that $F(\mathbf{a}, b) = 0$ and $\partial_y F(\mathbf{a}, b) \neq 0$. Then there exist positive numbers r_0, r_1 such that the following conclusions are valid.*

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a. *For each $\mathbf{x}$ in the ball $|\mathbf{x} - \mathbf{a}| < r_0$ there is a unique y such that $|y - b| < r_1$ and $F(\mathbf{x}, y) = 0$. We denote this y by $f(\mathbf{x})$; in particular, $f(\mathbf{a}) = b$.*

b. *The function f thus defined for $|\mathbf{x} - \mathbf{a}| < r_0$ is of class C^1, and its partial derivatives are given by*

$$\partial_j f(\mathbf{x}) = -\frac{\partial_j F(\mathbf{x}, f(\mathbf{x}))}{\partial_y F(\mathbf{x}, f(\mathbf{x}))}. \tag{3.2}$$

Already done part (a). So we have a function f such that

$F(x, f(x)) = 0$ on some neighborhood of a.

(b) Need to find $\partial_j f(x)$...

$$\partial_j f(x) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \boxed{\frac{f(x+he_j) - f(x)}{h}}$$

I know $F(x, f(x)) = 0$ and $F(x+he_j, f(x+he_j))$ supposing x and $x+he_j$ are in the neighborhood of a mentioned above.

$$F(x+he_j, f(x+he_j)) = F(x+he_j, \underbrace{\boxed{f(x+he_j) - f(x)}}_{k} + \underbrace{f(x)}_{y})$$

$$= F(x+he_j, y+k)$$

where $y = f(x)$ and $k = f(x+he_j) - f(x)$

$$0 = 0 - 0 = F(x+he_j, f(x+he_j)) - F(x, f(x))$$

$$= F(x+he_j, y+k) - F(x, y)$$

$$= F\big((x,y) + (he_j, k)\big) - F(x,y)$$

Let $\varphi(t) = F\big((x,y) + t(he_j,k)\big)$ then $\varphi: \mathbb{R} \to \mathbb{R}$

$$\varphi(1) - \varphi(0) = \varphi'(t)(1-0) \quad \text{for some } t \in (0,1)$$

$$F\big((x,y) - (he_j,k)\big) - F(x,y) = \nabla F\big((x,y) + t(he_j,k)\big) \cdot (he_j, k)$$

$$= \nabla F\big((x,y) + t(he_j,k)\big) \cdot (0, 0, \ldots, \underset{\substack{\uparrow \\ j\text{th slot.}}}{h}, \ldots 0, \; ,k)$$

$n+1$ partial

$$= h\,\partial_j F(x+the_j, y+tk) + k\,\partial_y F(x+the_j, y+tk) = 0$$

Now solve for k.

$$k = \frac{-h\,\partial_j F(x+the_j, y+tk)}{\partial_y F(x+the_j, y+tk)}$$

& by hypothesis $\partial_y F \neq 0$.

$$\partial_j f(x) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(x+he_j) - f(x)}{h} = \lim_{h \to 0} \frac{k}{h}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}\,\partial_j F(x+the_j, y+tk)}{\cancel{h}\,\partial_y F(x+the_j, y+tk)}$$

Since $F \in C^1$ then $\partial_j F$ and $\partial_y F$ are continuous...

Note also $k = f(x+he_j) - f(x) \to 0$ as $h \to 0$ since we already showed that f is cont.

$$\partial_j f(x) = \lim_{h \to 0} \frac{-h\,\partial_j F(x+the_j, y+tk)}{h\,\partial_y F(x+the_j, y+tk)} = \frac{-\partial_j F(x, f(x))}{\partial_y F(x, f(x))}$$

Finally to conclude f is of class C^1 note that F was of class C^1 and $\partial_y F(x, f(x)) \neq 0$ so this quotient $\frac{-\partial_j F(x, f(x))}{\partial_y F(x, f(x))}$ is continuous.

Thus, $\partial_j f(x)$ is continuous and so $f \in C^1$.

3.3 Corollary. *Let F be a function of class C^1 on $\mathbb{R}^n$, and let $S = \{\mathbf{x} : F(\mathbf{x}) = 0\}$. For every $\mathbf{a} \in S$ such that $\nabla F(\mathbf{a}) \neq \mathbf{0}$ there is a neighborhood N of $\mathbf{a}$ such that $S \cap N$ is the graph of a C^1 function.*

Proof. Since $\nabla F(\mathbf{a}) \neq \mathbf{0}$, we have $\boxed{\partial_j F(\mathbf{a}) \neq 0}$ for some j. The equation $F = 0$ can then be solved to yield x_j as a C^1 function of the remaining variables near the point $\mathbf{a}$. $\square$

This is the implicit function theorem with the j^{th} variable rather than the last one....

Example: $G(x,y) = x - e^{1-x} - y^3$

Consider the set $S = \{(x,y) : G(x,y) = 0\}$

$$\partial_x G(x,y) = 1 + e^{1-x} \qquad \partial_y G(x,y) = -\underbrace{3y^2}$$

if $y=0$ then $\partial_y G(x,y) = 0$

Suppose $G(x,y)=0$ and $\partial_y G(x,y) \neq 0$ then we know by the theorem that y can be written as a function of x.

$$G(x,y) = x - e^{1-x} - y^3 = 0$$

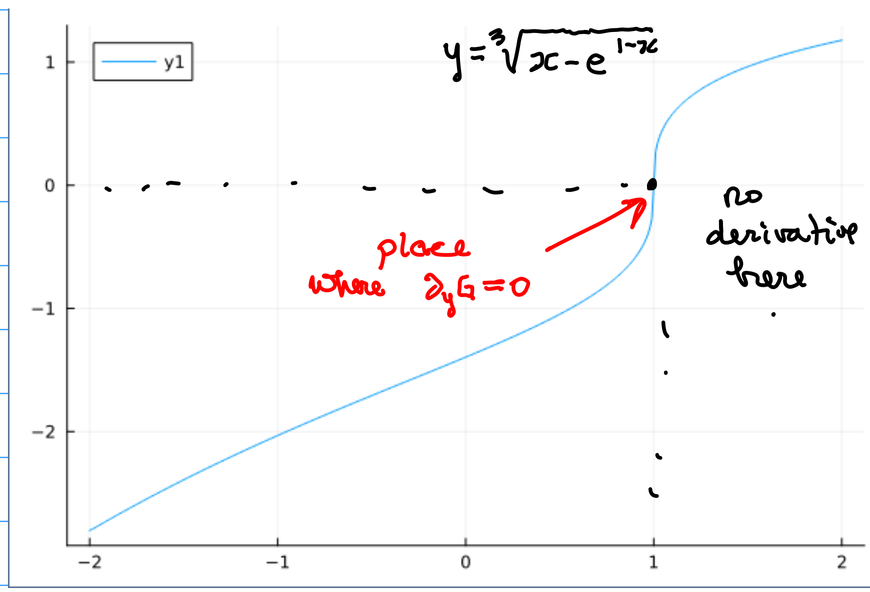
$$y^3 = x - e^{1-x}$$

In fact $y = \sqrt[3]{x - e^{1-x}}$ obviously true...

```
function f(x)
    r=x-exp(1-x)
    if r>=0
        return r^(1/3)
    else
        return -(-r)^(1/3)
    end
end
```

```
xs=-2:0.01:2
.01:2.0

plot(xs,f.(xs))
```



Suppose $G(x,y)=0$ and $\partial_x G(x,y) \neq 0$. Then the Corollary implies on some neighborhood the first coord x can be expressed as a function of y

$$G(x,y) = x - e^{1-x} - y^3 = 0$$

Solve x in terms of y...

$$\partial_x G(x,y) = 1 + e^{1-x} > 1 > 0$$

so there is a function $x = f(y)$ such that

$G(f(y), y) = 0$ in a neighborhood

$$f(y) - e^{1-f(y)} - y^3 = 0$$

3.2 Curves in the Plane

implicit function theorem

i. as the graph of a function, $y = f(x)$ or $x = f(y)$, where f is of class C^1;

easy

ii. as the locus[1] of an equation $F(x, y) = 0$, where F is of class C^1;

prove this... $\nabla F \neq 0$ on $\{(x,y) : F(x,y) = 0\}$

iii. easy parametrically, as the range of a C^1 function $\boxed{\mathbf{f}}: (a, b) \to \mathbb{R}^2$.

$f'(t) \neq 0$ for $t \in (a,b)$

Add to condions:

This suggests that it might be a good idea to impose the extra conditions that $\nabla F \neq \mathbf{0}$ on the set where $F = 0$ in (ii) and that $\mathbf{f}'(t) \neq \mathbf{0}$ in (iii). And indeed, with the help of the implicit function theorem, it is easy to see that under these extra conditions the representations (i)–(iii) are all *locally* equivalent. That is, if a curve is represented in one of the forms (i)–(iii) and $\mathbf{a}$ is a point on the curve, at least a small piece of the curve including the point $\mathbf{a}$ can also be represented in the other two forms.

Suppose $y = f(x)$ and $f \in C^1$ then

$F(x,y) = y - f(x)$ so $F(x,y) = 0$ implies $y - f(x) = 0$

note $F \in C^1$ so $y = f(x)$

if a curve can be described using (i)

then it can be described using (ii).

Suppose $y = f(x)$ and $f \in C^1$ then

$$\vec{f}(t) = (x, f(x))$$

then any point in the range of $\vec{f}(t)$ satisfies (i).

we go from (i) to (iii) way of describing a curve.

For next time Theorem 3.11

3.11 Theorem.

a. *Let F be a real-valued function of class C^1 on an open set in $\mathbb{R}^2$, and let $S = \{(x, y) : F(x, y) = 0\}$. If $\mathbf{a} \in S$ and $\nabla F(\mathbf{a}) \neq \mathbf{0}$, there is a neighborhood N of $\mathbf{a}$ in $\mathbb{R}^2$ such that $S \cap N$ is the graph of a C^1 function f (either $y = f(x)$ or $x = f(y)$).*

b. *Let $\mathbf{f} : (a, b) \to \mathbb{R}^2$ be a function of class C^1. If $\mathbf{f}'(t_0) \neq \mathbf{0}$, there is an open interval I containing t_0 such that the set $\{\mathbf{f}(t) : t \in I\}$ is the graph of a C^1 function f (either $y = f(x)$ or $x = f(y)$).*