3.9 Theorem (The Implicit Function Theorem for a System of Equations). *Let* $\mathbf{F}(\mathbf{x}, \mathbf{y})$ *be an* $\mathbb{R}^k$*-valued function of class* C^1 *on some neighborhood of a point* $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{n+k}$ *and let* $B_{ij} = (\partial F_i/\partial y_j)(\mathbf{a}, \mathbf{b})$. *Suppose that* $\mathbf{F}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$ *and* $\det B \neq 0$. *Then there exist positive numbers* r_0, r_1 *such that the following conclusions are valid.*

a. *For each* $\mathbf{x}$ *in the ball* $|\mathbf{x} - \mathbf{a}| < r_0$ *there is a unique* $\mathbf{y}$ *such that* $|\mathbf{y} - \mathbf{b}| < r_1$ *and* $\mathbf{F}(\mathbf{x}, \mathbf{y}) = 0$. *We denote this* $\mathbf{y}$ *by* $\mathbf{f}(\mathbf{x})$*; in particular,* $\mathbf{f}(\mathbf{a}) = \mathbf{b}$.

b. *The function* $\mathbf{f}$ *thus defined for* $|\mathbf{x} - \mathbf{a}| < r_0$ *is of class* C^1*, and its partial derivatives* $\partial_{x_j}\mathbf{f}$ *can be computed by differentiating the equations* $\mathbf{F}(\mathbf{x}, \mathbf{f}(\mathbf{x})) = \mathbf{0}$ *with respect to* x_j *and solving the resulting linear system of equations for* $\partial_{x_j} f_1, \ldots, \partial_{x_j} f_k$.

Note $(a,b) \in \mathbb{R}^{n+k}$ means $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^k$

$\mathcal{U} \subseteq \mathbb{R}^{n+k}$, $F : \mathcal{U} \to \mathbb{R}^k$

$$B = [B_{ij}] = \left[\frac{\partial F_i}{\partial y_j}(a,b)\right] = \begin{bmatrix} \partial F_1/\partial y_1 & \cdots & \partial F_1/\partial y_k \\ \vdots & & \vdots \\ \partial F_k/\partial y_1 & \cdots & \partial F_k/\partial y_k \end{bmatrix}$$

$F(a,b) = 0 \ldots$

$\exists r_0, r_1 : \forall x \in B(r_0, a)\ \exists y \in B(r_1, b)$ s.t. $F(x,y) = 0$

$f : B(r_0, a) \to B(r_1, b)$ note $B(r_0, a) \subseteq \mathbb{R}^n$ $B(r_1, b) \subseteq \mathbb{R}^k$

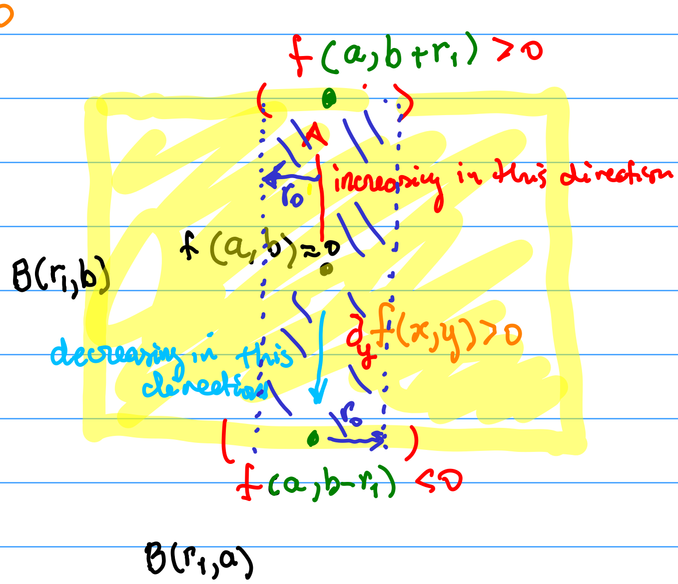
3.1 Theorem (The Implicit Function Theorem for a Single Equation). *Let $F(\mathbf{x}, y)$ be a function of class C^1 on some neighborhood of a point $(\mathbf{a}, b) \in \mathbb{R}^{n+1}$. Suppose that $F(\mathbf{a}, b) = 0$ and $\partial_y F(\mathbf{a}, b) \neq 0$. Then there exist positive numbers r_0, r_1 such that the following conclusions are valid.*

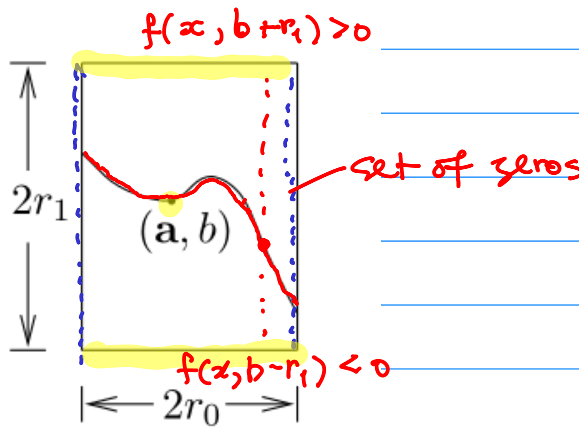
a. *For each $\mathbf{x}$ in the ball $|\mathbf{x} - \mathbf{a}| < r_0$ there is a unique y such that $|y - b| < r_1$ and $F(\mathbf{x}, y) = 0$. We denote this y by $f(\mathbf{x})$; in particular, $f(\mathbf{a}) = b$.*

b. *The function f thus defined for $|\mathbf{x} - \mathbf{a}| < r_0$ is of class C^1, and its partial derivatives are given by*

$$\partial_j f(\mathbf{x}) = -\frac{\partial_j F(\mathbf{x}, f(\mathbf{x}))}{\partial_y F(\mathbf{x}, f(\mathbf{x}))}. \tag{3.2}$$

$f(a,b)=0$





3.18 Theorem (The Inverse Mapping Theorem). *Let U and V be open sets in $\mathbb{R}^n$, $\mathbf{a} \in U$, and $\mathbf{b} = \mathbf{f}(\mathbf{a})$. Suppose that $\mathbf{f} : U \to V$ is a mapping of class C^1 and the Fréchet derivative $D\mathbf{f}(\mathbf{a})$ is invertible (that is, the Jacobian $\det D\mathbf{f}(\mathbf{a})$ is nonzero). Then there exist neighborhoods $M \subset U$ and $N \subset V$ of $\mathbf{a}$ and $\mathbf{b}$, respectively, so that $\mathbf{f}$ is a one-to-one map from M onto N, and the inverse map $\mathbf{f}^{-1}$ from N to M is also of class C^1. Moreover, if $\mathbf{y} = \mathbf{f}(\mathbf{x}) \in N$, $D(\mathbf{f}^{-1})(\mathbf{y}) = [D\mathbf{f}(\mathbf{x})]^{-1}$.*

$U, V \subseteq \mathbb{R}^n$ $\quad f : U \to V \quad a \in U$ and $Df(a) \in \mathbb{R}^{n \times n}$ is invertible.

open $\quad C^1(U)$

Let $F(x,y) = f(x) - y$ thus solving.

$y = f(x)$ is the same as $\boxed{F(x,y) = 0}$

use implicit fn. theorem to solve for x in terms of y and that's f^{-1}