$$F(x,y) = F(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_k)$$

Idea write $y = f(x)$ locally about (a,b) so that

$$F(x, f(x)) = 0$$ for all x in a neighborhood of a.

Assume we can do this for $k-1$.

$$F(x_1, x_2, \ldots, x_n, \underbrace{y_1, y_2, \ldots,}_{k-1} y_k)$$

extra

$$M^{kk} = \left. \begin{bmatrix} \partial F_1 / \partial y_1 & \cdots & \partial F_1 / \partial y_{k-1} \\ \vdots & \ddots & \vdots \\ \partial F_{k-1} / \partial y_1 & \cdots & \partial F_{k-1} / \partial y_{k-1} \end{bmatrix} \right|_{(x,y)=(a,b)}$$

Need $\det(M^{kk}) \neq 0$ ← in general $\det(M^{kk})$ might be zero. In that case relable the equations or variables...

By the induction hypothesis, $g : \mathbb{R}^{n+1} \to \mathbb{R}^{k-1}$ such that (y_k)

$$F_1(x, g(x, y_k), y_k) = 0$$
$$\vdots$$
$$F_{k-1}(x, g(x, y_k), y_k) = 0$$

for (x, y_k) in a neighborhood of (a, b_k).

Now we want to solve for y_k in terms of the x's so that

$$F_K(x, g(x,y_k), y_k) = 0$$

To do this we apply implicit function theorem for one equation.

I need $\left.\dfrac{\partial F_k(x, g(x,y_k), y_k)}{\partial y_k}\right|_{(x,y_k)=(a,b_k)} \neq 0$

$$\frac{\partial F_k(x,g(x,y_k),y_k)}{\partial y_k} = \sum_{j=1}^{k-1} \boxed{\frac{\partial F_k}{\partial y_j}}(x, g(x,y_k), y_k)\ \frac{\partial g_j(x,y_k)}{\partial y_k} + \boxed{\frac{\partial F_k}{\partial y_k}}(x, g(x,y_k), y_k)$$

$$\left.\frac{\partial F_k(x, g(x,y_k), y_k)}{\partial y_k}\right|_{(x,y_k)=(a,b_k)} = \sum_{j=1}^{k-1} B_{kj}\ \frac{\partial g_j}{\partial y_k}(a,b_k) + B_{kk}$$

need to write this

Use implicit differentiation to solve for $\partial g_j/\partial y_k$ from the $k-1$ equations that were used to obtain g.

Cramer's rule.

$$\partial g_j/\partial y_k = \frac{\det M_j^{kk}\left(-\begin{bmatrix} B_{1k} \\ \vdots \\ B_{k-1,k} \end{bmatrix}\right)}{\det M^{kk}}$$

replace the j^{th} column of M^{kk} with. $-\begin{bmatrix} B_{1k} \\ \vdots \\ B_{k-1,k} \end{bmatrix}$

$$
\begin{array}{c}
\hphantom{\begin{bmatrix} \partial F_1/\partial y_1 & \cdots & \end{bmatrix}} j^{th}\text{ column} \\
\left.\begin{bmatrix}
\partial F_1/\partial y_1 & \cdots & -\begin{bmatrix} B_{1k} \\ \vdots \\ B_{k-1,k} \end{bmatrix} & \cdots & \partial F_1/\partial y_{k-1} \\
\vdots & \ddots & & & \vdots \\
\partial F_{k-1}/\partial y_1 & \cdots & & \cdots & \partial F_{k-1}/\partial y_{k-1}
\end{bmatrix}\right|_{(x,y)=(a,b)}
\end{array}
$$

$$
= \begin{bmatrix}
B_{11} & \cdots & -B_{1k} & \cdots & B_{1,k-1} \\
\vdots & & & \underbrace{j+1,\ j+2,\ \ldots,\ k-1}_{k-j-1 \text{ columns here}} & \\
B_{k-1,1} & \cdots & -B_{k-1,k} & \cdots & B_{k-1,k-1}
\end{bmatrix}
$$

4 5 6 7

j^{th} column of B was replaced by the k^{th} column.

Compare the determinant of the above matrix to M^{kj}

last row and j^{th} column crossed out

$k-j-1$ sign changes because move the column to the end involves swapping columns $k-j-1$ times

1 more sign change because of the negative sign.

$$
\det \begin{bmatrix}
B_{11} & \cdots & -B_{1k} & \cdots & B_{1,k-1} \\
\vdots & & & \underbrace{j+1,\ j+2,\ \ldots,\ k-1}_{k-j-1 \text{ columns here}} & \\
B_{k-1,1} & \cdots & -B_{k-1,k} & \cdots & B_{k-1,k-1}
\end{bmatrix} = (-1)^{k-j} M^{kj}
$$

4 5 6 7

that were used to obtain g.

ramer's rule.

$$
\partial g_j/\partial y_k = \frac{\det M^{kk}_j\left(-\begin{bmatrix} B_{1k} \\ \vdots \\ B_{k-1,k} \end{bmatrix}\right)}{\det M^{kk}} = \frac{(-1)^{k-j} \det M^{kj}}{\det M^{kk}}
$$

$$\left.\frac{\partial F_k(x, g(x,y_k), \bar{y}_k)}{\partial y_k}\right|_{(x,y_k)=(a,b_k)} = \sum_{j=1}^{k-1} B_{kj} \frac{\partial g_j}{\partial y_k}(a, b_k) + B_{kk}$$

need to write this

$$= \sum_{j=1}^{k-1} B_{kj} \frac{(-1)^{k-j} \det M^{kj}}{\det M^{kk}} + B_{kk} \frac{(-1)^{k+k} \det M^{kk}}{\det M^{kk}}$$

$$= \frac{1}{\det M^{kk}} \sum_{j=1}^{k} B_{kj} (-1)^{k-j} \det M^{kj}$$

$$(-1)^{k-j} \cdot 1 = (-1)^{k-j} (-1)^{2j} = (-1)^{k+j}$$

By definition $\det B = \sum_{j=1}^{k} (-1)^{k+j} B_{kj} \det M^{kj}$

Therefore since $\det B \neq 0$ by hypothesis, then

$$\left.\frac{\partial F_k(x, g(x,y_k), \bar{y}_k)}{\partial y_k}\right|_{(x,y_k)=(a,b_k)} = \frac{\det B}{\det M^{kk}} \neq 0$$

By the implicit function theorem for one equation there is a $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $y_k = \varphi(x)$

$$F_k(x, g(x, \varphi(x)), \varphi(x)) = 0 \quad \text{in a neighborhood of } a$$

Now define $f : \mathbb{R}^n \to \mathbb{R}^k$ by

$$\begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_{k-1}(x) \\ f_k(x) \end{bmatrix} = \begin{bmatrix} g_1(x, \varphi(x)) \\ g_2(x, \varphi(x)) \\ \vdots \\ g_{k-1}(x, \varphi(x)) \\ \varphi(x) \end{bmatrix}$$

thus $y = f(x)$

Since g was obtain from the k-1 induction hypothesis then it's differentiable.

Since φ came from the theorem for one equation the it's differentiable.

Thus f is deferentiable (by the chain rule) and in all case one can use implicit differentiation to find the derivatives.

Integration ... review of 1 variable results...

a partition of $[a,b]$ is a (ordered) set $\{x_0, x_1, \dots, x_J\}$ such that $a = x_0 < x_1 < \cdots < x_J = b$.

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$$U(P,f) = \sum_{j=1}^{J} M_j \Delta x_j \quad = \quad S_P(f)$$

$M_j := \sup\{f(x) : x \in [x_{j-1}, x_j]\}$

$\Delta x_j := x_j - x_{j-1}$

$$L(P,f) = \sum_{j=1}^{J} m_j \Delta x_j \quad = \quad s_P(f)$$

$m_j := \inf\{f(x) : x \in [x_{j-1}, x_j]\}$