derivative of integral

4.15 Theorem (The Fundamental Theorem of Calculus).

large set

does $F(x)$ exist? small set $[a,x)$

a. *Let f be an integrable function on $[a,b]$. For $x \in [a,b]$, let $F(x) = \int_a^x f(t)\,dt$ (which is well defined by Theorem 4.9b). Then F is continuous on $[a,b]$; moreover, $F'(x)$ exists and equals $f(x)$ at every x at which f is continuous.*

integral of derivative ..

b. *Let F be a continuous function on $[a,b]$ that is differentiable except perhaps at finitely many points in $[a,b]$, and let f be a function on $[a,b]$ that agrees with F' at all points where the latter is defined. If f is integrable on $[a,b]$, then $\int_a^b f(t)\,dt = F(b) - F(a)$.*

Showing that $F'(x)$ exists and is equal to $f(x)$ every place where f is continuous. Let x be a point of continuity of f.

$$\left|\frac{F(y)-F(x)}{y-x} - f(x)\right| \leq \frac{1}{|y-x|}\left|\int_x^y |f(t)-f(x)|\,dt\right|$$

Suppose f is continuous at x. Fix x.

$\forall \varepsilon > 0\ \exists \delta > 0$ st $|t-x| < \delta$ implies $|f(t)-f(x)| < \varepsilon$

Now if $|y-x| < \delta$ then $|t-x| < \delta$ inside the integral.

Therefore

$$\left|\underbrace{\frac{F(y)-F(x)}{y-x}}_{\text{difference quotient for } F'.} - f(x)\right| < \frac{1}{|y-x|}\left|\int_x^y \varepsilon\,dt\right| = \frac{\cancel{|y-x|}\,\varepsilon}{\cancel{|y-x|}} = \varepsilon.$$

Therefore $\displaystyle\lim_{y\to x}\frac{F(y)-F(x)}{y-x} = f(x)$ where x is a point of continuity of f.

Let F be a continuous function on $[a,b]$ that is differentiable except perhaps at finitely many points in $[a,b]$, and let f be a function on $[a,b]$ that agrees with F' at all points where the latter is defined. If f is integrable on $[a,b]$, then $\int_a^b f(t)\,dt = F(b) - F(a)$.

hypothesis

Note: $f(x) = F'(x)$ when F' exists.

Proof Since $f \in \mathcal{R}[a,b]$ then for any $\varepsilon > 0$ there is a partitions P such that

$$|s_P(f) - S_P(f)| < \varepsilon$$

The points where $F'(x)$ doesn't exist are finite and let's add them to $\tilde{P} = P \cup \{\underbrace{\text{places where } F' \text{ doesn't exist}}_{\text{finite}}\}$

$$\tilde{P} = \{x_0, x_1, \dots, x_J\} \qquad a = x_0 < x_1 < \cdots < x_J = b$$

$$s_P(f) \le s_{\tilde{P}}(f) \quad \text{and} \quad S_{\tilde{P}}(f) \le S_P(f)$$

Note that F is differentiable on (x_{j-1}, x_j) for every j

By the mean value theorem

$$F(x_j) - F(x_{j-1}) = F'(t_j)(x_j - x_{j-1}) = f(t_j)(x_j - x_{j-1})$$

for some $t_j \in (x_{j-1}, x_j)$

Sum over j

$$F(a) - F(b) = \sum_{j=1}^{J} F(x_j) - F(x_{j-1}) = \underbrace{\sum_{j=1}^{J} f(t_j)(x_j - x_{j-1})}_{\text{Reimann sum.}}$$

$$m_j = \inf\{f(t) : t \in [x_{j-1}, x_j]\} \le f(t_j) \le \sup\{f(t) : t \in [x_{j-1}, x_j]\} = M_j$$

Recall $\quad s_{\tilde{P}}(f) = \sum_{j=1}^{J} m_j (x_j - x_{j-1}), \quad S_{\tilde{P}}(f) = \sum_{j=1}^{J} M_j (x_j - x_{j-1})$

Therefore

$$s_{\tilde{P}}(f) \le \underbrace{F(a) - F(b)} \le S_{\tilde{P}}(f)$$

Since $\left| s_P(f) - S_P(f) \right| < \varepsilon$ then $\left| s_{\tilde{P}}(f) - S_{\tilde{P}}(f) \right| < \varepsilon$

Thus $$\left| \underbrace{F(a) - F(b)} - \int_a^b f \right| < \varepsilon$$

and since ε was arbitrary $\underbrace{F(a) - F(b)} = \int_a^b f$.

Let

$$R = [a,b] \times [c,d] = \{(x,y) \in \mathbb{R}^2 : x \in [a,b] \text{ and } y \in [c,d]\}$$

and partition the rectangle into smaller rectangles

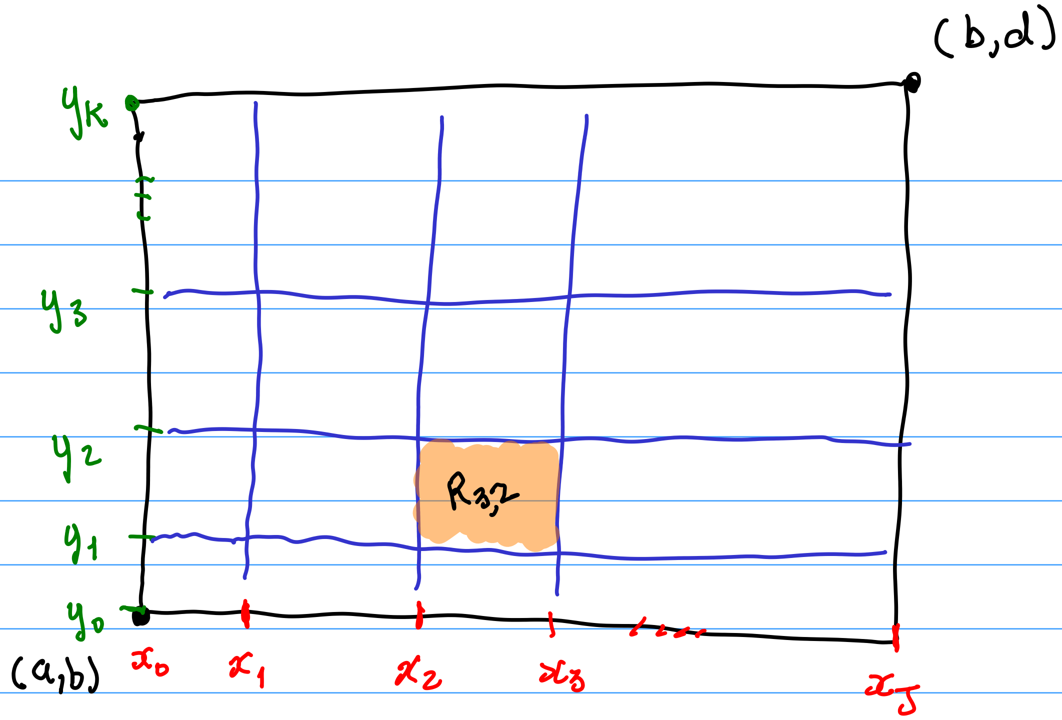
$$P = \{\underbrace{x_0, x_1, \dots, x_J}_{\text{partition of } [a,b]} ; \underbrace{y_0, y_1, \dots, y_K}_{\text{partition of } [c,d]}\}$$

$a = x_0 < x_1 < \dots < x_J = b$

$c = y_0 < y_1 < \dots < y_K = d$

Let $R_{jk} = [x_{j-1}, x_j] \times [y_{k-1}, y_k]$

Area of rectangle $\quad |R_{jk}| = \underbrace{(x_j - x_{j-1})}_{\text{width}} \underbrace{(y_k - y_{k-1})}_{\text{height}}$



$$M_{jk} = \sup \{ f(x,y) : (x,y) \in R_{jk} \}$$

$$m_{jk} = \inf \{ f(x,y) : (x,y) \in R_{jk} \}$$

$$s_p(f) = \sum_{j=1}^{J} \sum_{k=1}^{K} m_{jk} |R_{jk}| = \sum_{k=1}^{K} \sum_{j=1}^{J} m_{jk} |R_{jk}|$$

order doesn't matter because there are finite sums

$$S_p(f) = \sum_{j=1}^{J} \sum_{k=1}^{K} M_{jk} |R_{jk}|$$

Upper and lower sum:

$$s_p(f) = \sum_{j,k} m_{jk} |R_{jk}| \qquad S_p(f) = \sum_{j,k} M_{jk} |R_{jk}|$$

$$\underline{I}_R(f) = \sup \{ s_p(f) : \text{where } p \text{ is a partition of } R \}$$

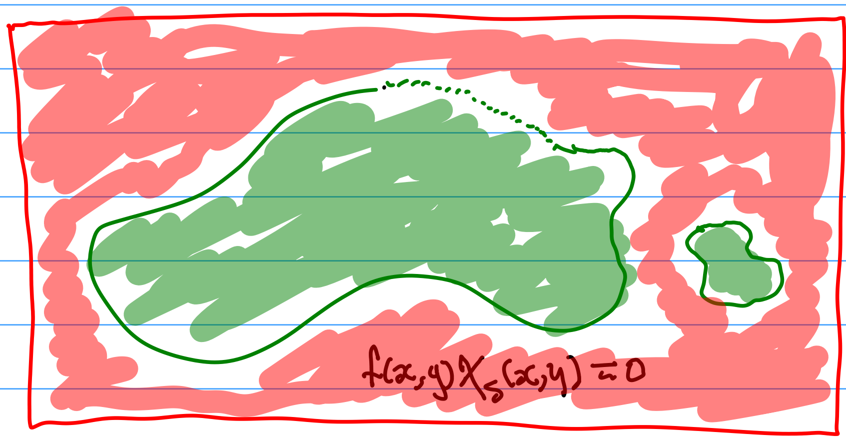
$$\overline{I}_R(f) = \inf \{ S_p(f) : \text{where } p \text{ is a partition of } R \}$$

If $\underline{I}_R(f) = \overline{I}_R(f)$ then we say f is Reimann integrable and $\iint_R f \, dA$ is that common value.

How to adapt proofs so this all works out?

Integral over an interval was general enough in $\mathbb{R}$.
But in $\mathbb{R}^2$ the sets over which one might want to integrate come in many shapes.

Consider a bounded $S \subseteq \mathbb{R}^2$, what is $\iint_S f(x,y)\, dA$?



since it's bounded it fits inside a rectangle.

Characteristic function of S $\quad \chi_S(x,y) = \begin{cases} 1 & \text{if } (x,y) \in S \\ 0 & \text{otherwise.} \end{cases}$
$\chi_S : \mathbb{R}^2 \to \mathbb{R}$

So
$$f(x,y)\chi_S(x,y) = \begin{cases} f(x,y) & \text{for } (x,y) \in S \\ 0 & \text{for } (x,y) \notin S. \end{cases}$$

Define
$$\iint_S f\, dA = \iint_R f \chi_S$$

$\therefore$ If we want to integrate $\iint_S f\, dA$ when f is continuous then we have to integrate $\iint_R f\chi_S$ where $f\chi_S$ is not continuous...

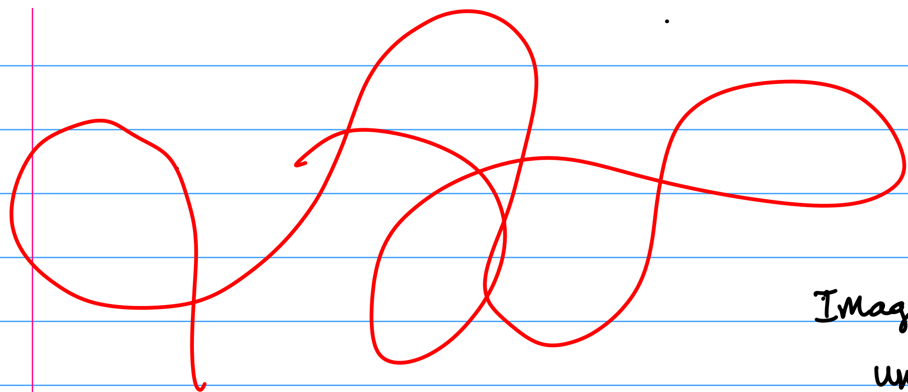
The starting point is the analogue of Theorem 4.13. The notion of “zero content” transfers readily to sets in the plane; namely, a set $Z \subset \mathbb{R}^2$ is said to have **zero content** if for any $\epsilon > 0$ there is a finite collection of rectangles $R_1, \ldots, R_M$ such that (i) $Z \subset \bigcup_1^M R_m$, and (ii) the sum of the areas of the R_m's is less than ϵ. We then have:

4.19 Proposition.

a. *If $Z \subset \mathbb{R}^2$ has zero content and $U \subset Z$, then U has zero content.*

b. *If $Z_1, \ldots, Z_k$ have zero content, then so does $\bigcup_1^k Z_j$.*

c. *If $\mathbf{f} : (a_0, b_0) \to \mathbb{R}^2$ is of class C^1, then $\mathbf{f}([a, b])$ has zero content whenever $a_0 < a < b < b_0$.*



has zero content in $\mathbb{R}^2$

Image of a closed interval under a C^1 function.