Let $g \in C^1([a,b])$ be 1-to-1

$f : \mathbb{R} \to \mathbb{R}$ is continuous...

$$\int_a^b f(g(u))\, g'(u)\, du = \int_{g(a)}^{g(b)} f(x)\, dx.$$

2.26 Theorem (Chain Rule I). *Suppose that* $\mathbf{g}(t)$ *is differentiable at* $t = a$, $f(\mathbf{x})$ *is differentiable at* $\mathbf{x} = \mathbf{b}$, *and* $\mathbf{b} = \mathbf{g}(a)$. *Then the composite function* $\varphi(t) = f(\mathbf{g}(t))$ *is differentiable at* $t = a$, *and its derivative is given by*

$$\varphi'(a) = \nabla f(\mathbf{b}) \cdot \mathbf{g}'(a),$$

4.15 Theorem (The Fundamental Theorem of Calculus).

a. *Let* f *be an integrable function on* $[a, b]$. *For* $x \in [a, b]$, *let* $F(x) = \int_a^x f(t)\, dt$ *(which is well defined by Theorem 4.9b). Then* F *is continuous on* $[a, b]$; *moreover,* $F'(x)$ *exists and equals* $f(x)$ *at every* x *at which* f *is continuous.*

b. *Let* F *be a continuous function on* $[a, b]$ *that is differentiable except perhaps at finitely many points in* $[a, b]$, *and let* f *be a function on* $[a, b]$ *that agrees with* F' *at all points where the latter is defined. If* f *is integrable on* $[a, b]$, *then* $\int_a^b f(t)\, dt = F(b) - F(a)$.

$F \circ g$

$$\int_a^b (F\circ g)'(u)\,du = (F\circ g)(b) - (F\circ g)(a)$$

Since f is continuous then f is integrable so by Thm. 4.15 a

$F(t) = \int_{g(a)}^{t} f(x)\,dx$ is well defined and F is cont and F' exists and $\boxed{F'(x) = f(x)}$ everywhere since f was continuous...

Consider $(F\circ g)(u)$ since F is differentiable and g was differentiable by assumption, thus by Theorem 2.26

$$(F\circ g)'(u) = F'(g(u))\, g'(u)$$

Since $F(g(a)) = 0$ then

was just 0.

$$\int_{g(a)}^{g(b)} f(x)\,dx = F(g(b)) = F(g(b)) - F(g(a)).$$

Same

By theorem 4.15b applied to $F \circ g$ we have

$$\int_a^b (F\circ g)'(u)\,du = (F\circ g)(b) - (F\circ g)(a) = F(g(b)) - F(g(a))$$

Therefore

$$\int_{g(a)}^{g(b)} f(x)\,dx = \int_a^b (F\circ g)'(u)\,du = \int_a^b F'(g(u))\,g'(u)\,du$$

again 4.15a with $\boxed{F'(x) = f(x)}$

Thus $$\int_{g(a)}^{g(b)} f(x)\,dx = \int_a^b f(g(u))\,g'(u)\,du$$

How to generalize to multi-dimensional integrals.

① Extend the 1-dimensional proof ☑ mostly working

② Find a new proof ☑ today's approach ...

How do simpler transformations affect the integral and then try to build a general differentiable coordinate transformation g out of simpler things...

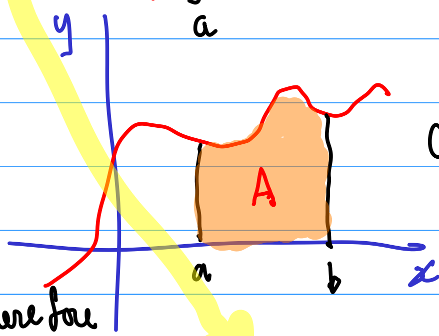
Let $S = g([a,b])$ then $[a,b] = g^{-1}(S)$

$$\int_{\substack{S \\ = [a,b]}} f(x)\,dx = \int_{\substack{g^{-1}(S) \\ = [2a,2b]}} f(g(u))\,|g'(u)|\,du$$

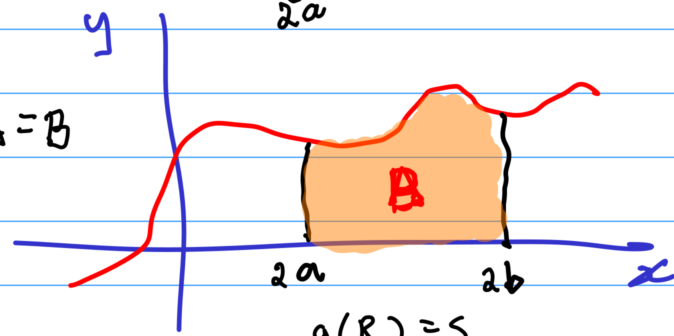
$g(x) = \frac{x}{2}$ $\qquad g^{-1}(y) = 2y$ $\qquad g^{-1}([a,b]) = [2a, 2b]$

In one dimension

$$A = \int_a^b f(x)\,dx \qquad\qquad B = \int_{2a}^{2b} f\left(\tfrac{x}{2}\right) dx$$



Clearly $2A = B$



Therefore

$g(R) = S$

$g([2a, 2b]) = [a, b]$.

$g(x) = \frac{x}{2}$ so $g'(x) = \frac{1}{2}$

$$\int_a^b f(x)\,dx = \frac{1}{2}\int_{2a}^{2b} f\left(\tfrac{x}{2}\right) dx = \int_{2a}^{2b} f\left(\tfrac{x}{2}\right)\tfrac{1}{2}\,dx = \int_{2a}^{2b} f(g(u))\,g'(u)\,du$$

We do this same argument in n dimensions taking

$G(x) = Ax$ where $A \in \mathbb{R}^{n \times n}$ assume A^{-1} exists

$DG(x) = A$

General setup.

assume $\begin{cases} G : \mathbb{R}^n \to \mathbb{R}^n \\ G(R) = S \\ G \text{ is one-to-one on } R \\ G \in C'(R). \end{cases}$

Let f be integrable.

Theorem:

prove this term is correct

$$(*) \qquad \int_S \cdots \int f(x)\,d^n x = \int_{G^{-1}(S)} \cdots \int f(G(u))\,\left|\det DG(u)\right|\,d^n u$$

If. $G(x) = Ax$ then $DG(x) = A$ and

$$\int\cdots\int_S f(x)\, d^n x = \int\cdots\int_{A^{-1}(S)} f(Au)\, |\det A|\, d^n u$$

As the first step to prove $(*)$ for general $G \in C^1(R)$ we first show in the case $G(x) = Ax$ and $A \in \mathbb{R}^{n\times n}$ invertible.

Recall linear algebra... If A is invertible one can find A^{-1} using the augmented matrix

$$[\, A \mid I \,]$$

identity matrix.

and the performing elementary row operations until

$$[\, I \mid A^{-1} \,].$$

What this really does is factor A into a product of elementary matrices corresponding to

1. Multiply the kth component by a nonzero number c, $\quad r_k \leftarrow c\, r_k$ (row k)

   rescaling

2. Add a multiple of the jth component to the kth component, $\quad r_k \leftarrow r_k + c\, r_j$

   elimination step

3. Interchange the jth and kth components: $\quad r_k \leftrightarrow r_j$

   row swap

Use composition of row operations with identity matrix to find the matrix corresponding to the row operation