Next week start with Quiz instead of homework.

I'll choose some HW problems and quiz will cover those.

B.9 Theorem. *Let $R = [a, b] \times [c, d]$, and let f be an integrable function on R. Suppose that, for each $y \in [c, d]$, the function f_y defined by $f_y(x) = f(x, y)$ is integrable on $[a, b]$, and the function $g(y) = \int_a^b f(x, y)\, dx$ is integrable on $[c, d]$. Then*

$$\iint_R f\, dA = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy.$$

$\left| \int_c^d g(y)dy - \sum_{k=1}^{K} g(y_k)\Delta y_k \right| < \frac{\varepsilon}{3}$

Proof. Let $P_{JK} = \{x_0, \ldots, x_J; y_0, \ldots, y_K\}$ be the partition of R obtained by subdividing $[a, b]$ and $[c, d]$, respectively, into J and K equal subintervals of length $\Delta x = (b - a)/J$ and $\Delta y = (d - c)/K$. Given $\epsilon > 0$, there is an integer N such that

$$\text{(B.10)} \qquad \boxed{\left| \iint_R f\, dA - \sum_{j=1}^{J} \sum_{k=1}^{K} f(x_j, y_k)\, \Delta x\, \Delta y \right| < \frac{\epsilon}{3}}$$

provided that $J \geq N$ and $K \geq N$, and also

$$\text{(B.11)} \qquad \boxed{\left| \int_c^d \underbrace{\left[\int_a^b f(x, y)\, dx \right]}_{g(y)} dy - \sum_{k=1}^{K} \underbrace{\int_a^b f(x, y_k)\, dx}_{g(y_k)}\, \Delta y \right| < \frac{\epsilon}{3}}$$

provided that $K \geq N$. (For (B.10) we are applying Theorem B.8 to the function f, and for (B.11) we are applying Theorem B.7 to the function $g(y) = \int_a^b f(x, y)\, dx$.)

Let us fix K to be equal to N; then the points y_k are also fixed. By Theorem B.7 again, we can choose J large enough so that (and there is a finite # of them)

$$\boxed{\left| \int_a^b f(x, y_k)\, dx - \sum_{j=1}^{J} f(x_j, y_k)\, \Delta x \right| < \frac{\epsilon}{3(d - c)}}$$

for all $k = 1, \ldots, K$. Then

finite number of J_k

$f_y(x) = f(x, y)$ is integrable of $[a, b]$ for any y fixed.

For each $y = y_k$ we can find a partition P_k fine enough

i.e. J_k large enough so that

$$\left|\int_a^b f(x, y_k)\,dx - \sum_{j=1}^{J_k} f(x_j, y_k)\,\Delta x\right| < \frac{\epsilon}{3(d-c)}$$

Now let $J = \max\{J_k : k = 1, 2, \ldots, K\}$.

$$\left|\sum_{j=1}^{J}\sum_{k=1}^{K} f(x_j, y_k)\,\Delta x\,\Delta y - \sum_{k=1}^{K}\int_a^b f(x, y_k)\,dx\,\Delta y\right|$$

$$\leq \sum_{k=1}^{K}\left|\sum_{j=1}^{J} f(x_j, y_k)\,\Delta x - \int_a^b f(x, y_k)\,dx\right|\Delta y < \frac{K\epsilon\Delta y}{3(d-c)} = \frac{\epsilon}{3}.$$

Therefore, by (B.10),

$$\left|\iint_R f\,dA - \sum_{k=1}^{K}\int_a^b f(x, y_k)\,dx\,\Delta y\right| < \frac{2\epsilon}{3},$$

and hence by (B.11),

$$\left|\iint_R f\,dA - \int_c^d\left[\int_a^b f(x, y)\,dx\right]dy\right| < \epsilon.$$

Since ϵ is arbitrary, the double integral and the iterated integral must be equal. $\square$

$$\left|\iint_R f\,dA - \int_c^d\left[\int_a^b f(x, y)\,dx\right]dy\right|$$

$$\leq \left|\iint_R f\,dA - \sum_{j=1}^{J}\sum_{k=1}^{K} f(x_j, y_k)\,\Delta x\,\Delta y\right| + \left|\sum_{j=1}^{J}\sum_{k=1}^{K} f(x_j, y_k)\,\Delta x\,\Delta y\right.$$

$$\left. - \sum_{k=1}^{K}\int_a^b f(x, y_k)\,dx\,\Delta y\right| + \left|\sum_{k=1}^{K}\int_a^b f(x, y_k)\,dx\,\Delta y - \int_c^d\left[\int_a^b f(x, y)\,dx\right]dy\right|$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

$$\int\cdots\int_S f(\mathbf{x})\, d^n\mathbf{x} = \underbrace{\int\int}_{\text{Some large rectangle that contains } S.} \chi_S(x) f(x)\, d^n x = \underbrace{\int\cdots\int}_{\text{Some large rectangle that contains } S.} g(x)\, d^n x$$

Let $g(x) = \chi_S(x) f(x)$

$$\underbrace{\int\cdots\int}_{\text{Some large rectangle that contains } S.} g(x)\, d^n x = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g(x)\, d^n x$$

iterated integral

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\cdots \int_{-\infty}^{\infty} g(x_1, x_2, \dots x_n)\, dx_n \cdots\right) dx_2\right) dx_1$$

looks like improper integral, but not ... just sufficiently large rectangle so $g(x)=0$ outside that rectangle

integrands vanish outside bounded sets. The point is that now we don't have to worry about what the limits of integration in each variable are; we can take them to be $\pm\infty$.

Step 2. We prove the theorem when G is an "elementary transformation," that

row operations act on 1 or 2 variables at a time, so only look at the relevant 1-dim. iterated integrals to see what happens.

Let's look at the elimination step...

$r_k \leftarrow r_k + c r_j$.

elimination

as in the discussion preceding (4.34).) Likewise, for $\mathbf{G}_2$ we set $x_k = u_k + cu_j$ and obtain

$$\int_{-\infty}^{\infty} f(\ldots, x_k, \ldots)\, dx_k = \int_{-\infty}^{\infty} f(\ldots, u_k + cu_j, \ldots)\, du_k.$$

constant with respect to u_k

(u_j is a *constant* as far as this calculation is concerned.) Now an integration with

shifting doesn't change the area...

finished the proof of ?

4.37 Theorem. *Let A be an invertible $n \times n$ matrix, and let $\mathbf{G}(\mathbf{u}) = A\mathbf{u}$ be the corresponding linear transformation of $\mathbb{R}^n$. Suppose S is a measurable region in $\mathbb{R}^n$ and f is an integrable function on S. Then $\mathbf{G}^{-1}(S) = \{A^{-1}\mathbf{x} : \mathbf{x} \in S\}$ is measurable and $f \circ \mathbf{G}$ is integrable on $\mathbf{G}^{-1}(S)$, and*

$$\text{(4.38)} \qquad \int \cdots \int_S f(\mathbf{x})\, d^n\mathbf{x} = |\det A| \int \cdots \int_{\mathbf{G}^{-1}(S)} f(A\mathbf{u})\, d^n\mathbf{u}.$$

General result...

4.41 Theorem. *Given open sets U and V in $\mathbb{R}^n$, let $\mathbf{G} : U \to V$ be a one-to-one transformation of class C^1 whose derivative $D\mathbf{G}(\mathbf{u})$ is invertible for all $\mathbf{u} \in U$. Suppose that $T \subset U$ and $S \subset V$ are measurable sets such that $\overline{T} \subset U$ and $\mathbf{G}(T) = S$. If f is an integrable function on S, then $f \circ \mathbf{G}$ is integrable on T, and*

$$\text{(4.42)} \qquad \int \cdots \int_S f(\mathbf{x})\, d^n\mathbf{x} = \int \cdots \int_T f(\mathbf{G}(\mathbf{u}))|\det D\mathbf{G}(\mathbf{u})|\, d^n\mathbf{u}.$$

Intuitively work with differentials

$$G(u+du) = \underbrace{G(u)}_{\text{shift}} + \underbrace{DG(u)}_{\text{matrix mult.}}\, du$$

Maybe like

$$G(u) = b + Au$$ ← linear function plus shift.

Still holds

$G(u) = b + Au$

4.37 Theorem. *Let A be an invertible $n \times n$ matrix, and let* $\cancel{G(u) = Au}$ *be the corresponding linear transformation of $\mathbb{R}^n$. Suppose S is a measurable region in $\mathbb{R}^n$ and f is an integrable function on S. Then $\mathbf{G}^{-1}(S) = \{A^{-1}\mathbf{x} : \mathbf{x} \in S\}$ is measurable and $f \circ \mathbf{G}$ is integrable on $\mathbf{G}^{-1}(S)$, and*

$$\int \cdots \int_S f(\mathbf{x})\, d^n\mathbf{x} = |\det A| \int \cdots \int_{\mathbf{G}^{-1}(S)} f(A\mathbf{u})\, d^n\mathbf{u}. \tag{4.38}$$

$$G(u+du) = \underbrace{G(u)}_{\text{shift}} + \underbrace{DG(u)}_{\text{matrix mult.}} du$$

4.41 Theorem. *Given open sets U and V in $\mathbb{R}^n$, let $\mathbf{G} : U \to V$ be a one-to-one transformation of class C^1 whose derivative $D\mathbf{G}(\mathbf{u})$ is invertible for all $\mathbf{u} \in U$. Suppose that $T \subset U$ and $S \subset V$ are measurable sets such that $\overline{T} \subset U$ and $\mathbf{G}(T) = S$. If f is an integrable function on S, then $f \circ \mathbf{G}$ is integrable on T, and*

$$\int \cdots \int_S f(\mathbf{x})\, d^n\mathbf{x} = \int \cdots \int_T f(\mathbf{G}(\mathbf{u}))|\det D\mathbf{G}(\mathbf{u})|\, d^n\mathbf{u}. \tag{4.42}$$

put the $|\det DG(u)|$ inside the integral since u is changing ... this is addressed in the

in Appendix B.5 (Theorem B.24),

Chapter 5

Length of a curve:

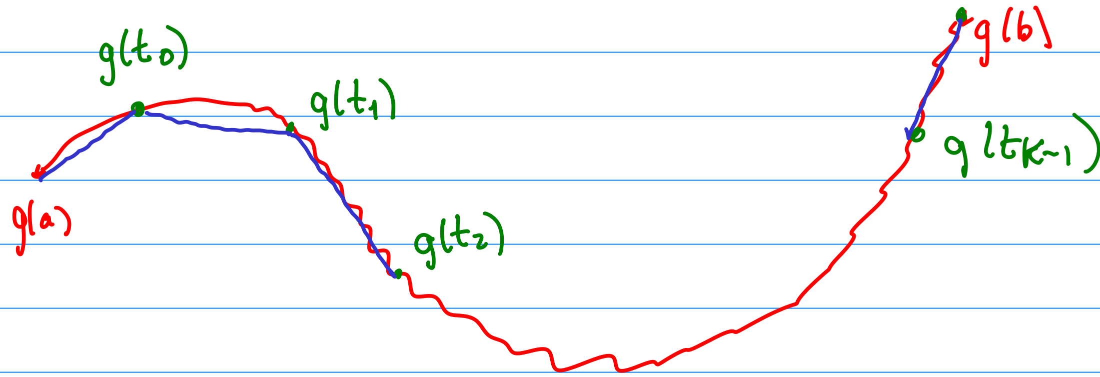
Let $g : [a,b] \to C$ be continuous parameterization of the curve

$$L_p(C) = \sum |g(t_k) - g(t_{k-1})| \quad \text{where } P = \{t_0, \cdots t_K\} \text{ is a partition of } [a,b]$$

supremum over all partitions

$$L(C) = \sup\{L_p(C) : p \text{ is a partition}\}$$

$a = t_0 < t_1 < t_2 \cdots < t_K = b$



$$L_P(C) = \sum_{k=1}^{K} \left| g(t_k) - g(t_{k-1}) \right|$$

Supremum $L(C)$ always exists in $\mathbb{R} \cup \{\infty\}$, but is it finite? For what curves is it finite?