Changing the parameterization in

$$\int_C F\cdot dx$$

depends on the orientation of the parameterization.

$$\int_C F\cdot dx = \int_{[a,b]} F(g(t))\cdot g'(t)\,dt$$

$$\int_I f(x)\,dx = \int_{\varphi^{-1}(I)} f(\varphi(u))|\varphi'(u)|\,du.$$

New parameterization $h:[c,d]\to C$

$h(u) = g(\varphi(u))$

$\varphi:[c,d]\to[a,b]$

know φ is 1-to-1 and C'

remove abs value here

$$\int_{[a,b]} F(g(t))\cdot g'(t)\,dt = \int_{\underset{[c,d]}{\varphi^{-1}([a,b])}} F(g(\varphi(u)))\cdot \underbrace{g'(\varphi(u))}_{\text{no abs value here}}|\varphi'(u)|\,du$$

Since φ is 1-to-1 and C'

then either $\varphi'(u)\geq 0$ for all $u\in[c,d]$

or $\varphi'(u)\leq 0$ for all $u\in[c,d]$

$$t = \int_{\varphi^{-1}([a,b])} f(g(\varphi(u)))\,\underbrace{|g'(\varphi(u))|\,|\varphi'(u)|}_{\text{both inside abs value}}\,du$$

put together

$$\text{))}\,|\underbrace{g'(\varphi(u))\,\varphi'(u)}_{\text{chain rule.}}|\,du$$

Case φ is increasing and

$\varphi'(u)\geq 0$ for all $u\in[c,d]$

$$\int_{[c,d]} F(g(\varphi(u)))\cdot g'(\varphi(u))\,|\varphi'(u)|\,du$$

$$= \int_{[c,d]} F(g(\varphi(u)))\cdot \underbrace{g'(\varphi(u))\,\varphi'(u)}_{\text{chain rule.}}\,du$$

$$= \int_{[c,d]} F(g\circ\varphi(u))\cdot (g\circ\varphi)'(u)\, du$$

Therefore

$$\int_{[a,b]} F(g(t))\cdot g'(t)\, dt = \int_{[c,d]} F(h(u))\cdot h'(u)\, du$$

So the line integral of a vector valued function doesn't depend on the parameterization provided the orientation doesn't change.

Case φ is decreasing and
$\varphi'(u) \leq 0$ for all $u \in [c,d]$

$$\int_{[c,d]} F(g(\varphi(u)))\cdot g'(\varphi(u))\,|\varphi'(u)|\, du$$

$$= \int_{[c,d]} F(g(\varphi(u)))\cdot \underbrace{g'(\varphi(u))(-\varphi'(u))}_{\text{Chain rule.}}\, du$$

$$= -\int_{[c,d]} F(h(u))\cdot h'(u)\, du$$

So if the orientation of the parameterization changes then there is an additional minus sign.

Next up Green's Theorem...

Definition

A **simple closed curve** in $\mathbb{R}^n$ is a curve whose starting and ending points coincide, but that does not intersect itself otherwise. More precisely, a simple closed curve is one that can be parametrized by a continuous map $\mathbf{x} = \mathbf{g}(t)$, $a \le t \le b$, such that $\mathbf{g}(a) = \mathbf{g}(b)$ but $\mathbf{g}(s) \neq \mathbf{g}(t)$ unless $\{s, t\} = \{a, b\}$. or $s=t$

what does this mean...

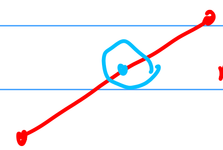
want g to be 1-to-1 on $[a, b)$
and $g(a) = g(b)$

$S = \overline{S^{int}}$ and ~~compact~~ bounded.

Definition

We shall use the term **regular region** to mean a compact set in $\mathbb{R}^n$ that is the closure of its interior. Equivalently, a compact set $S \subset \mathbb{R}^n$ is a regular region if every neighborhood of every point on the boundary ∂S contains points in S^{int}. For example, a closed ball is a regular region, but a closed line segment in $\mathbb{R}^n$ $(n > 1)$ is not, because its interior is empty.

compact is closed and bounded



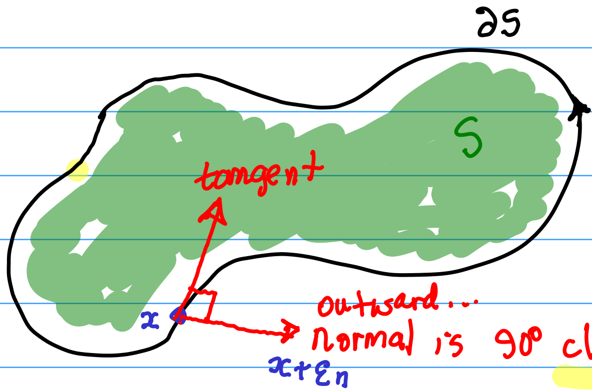
no points in the interior of this point on the boundary because there is no interior at all.

Now let $n = 2$. We say that a regular region $S \subset \mathbb{R}^2$ has a **piecewise smooth boundary** if the boundary ∂S consists of a finite union of disjoint, piecewise smooth simple closed curves, where "piecewise smooth" has the meaning assigned in the previous section. (We thus allow the possibility that S contains "holes," so

The notion of arc length extends in an obvious way to **piecewise smooth** curves, obtained by joining finitely many smooth curves together end-to-end but allowing corners or cusps at the joining points; we simply compute the lengths of the smooth pieces and add them up. We can express this more precisely in terms of parametrizations, as follows: The function $\mathbf{g} : [a, b] \to \mathbb{R}^n$ is called **piecewise smooth** if (i) it is continuous, and (ii) its derivative exists and is continuous except perhaps at finitely many points t_j where the one-sided limits $\lim_{t \to t_j \pm} \mathbf{g}'(t)$ exist.

pg 214

that its boundary may be disconnected.) In this case, the **positive orientation** on ∂S is the orientation on each of the closed curves that make up the boundary such that the region S is on the *left* with respect to the positive direction on the curve. More precisely, if $\mathbf{x}$ is a point on ∂S at which ∂S is smooth, and $\mathbf{t} = (t_1, t_2)$ is the unit tangent vector in the positive direction at that point, then the vector $\mathbf{n} = (t_2, -t_1)$, obtained by rotating $\mathbf{t}$ by 90° *clockwise*, points *out* of S. (That is, $\mathbf{x} + \epsilon\mathbf{n} \notin S$ for small $\epsilon > 0$.) See Figure 5.4.



orientation move counter clockwise around the region..

outward... normal is 90° clockwise from the tangent

for every $x \in \partial S$ there is $\delta > 0$ such that $x + \varepsilon n \notin S$ for all $0 < \varepsilon < \delta$.

We were just discussing the line integral

$$\int_C F \cdot dx = \int_{[a,b]} F(g(t)) \cdot g'(t)\,dt$$

$$\updownarrow$$

$$\underbrace{\int_{\partial S} F_1\,dx_1 + F_2\,dx_2}_{\text{notation of differential forms} \sim \text{1-form..}}$$ one more notation...

S compact and $S = \overline{S^{int}}$

5.12 Theorem (Green's Theorem). *Suppose S is a regular region in $\mathbb{R}^2$ with piecewise smooth boundary ∂S. Suppose also that $\mathbf{F}$ is a vector field of class C^1 on $\overline{S}$. Then* parameterized by piecewise smooth curve...

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{x} = \iint_S \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) dA. \tag{5.13}$$

In the more common notation, if we set $\mathbf{F} = (P, Q)$ and $\mathbf{x} = (x, y)$,

$$\boxed{\int_{\partial S} P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.} \tag{5.14}$$

Stronger hypothesis: suppose S is x-simple and y-simple (at the same time).

quite simple. We shall say that the region S is **x-simple** if it is the region between the graphs of two functions of x, that is, if it has the form

$$S = \{(x, y) : a \le x \le b,\ \varphi_1(x) \le y \le \varphi_2(x)\}, \tag{5.15}$$

where φ_1 and φ_2 are continuous, piecewise smooth functions on $[a, b]$. Likewise, we say that S is **y-simple** if it has the form

$$S = \{(x, y) : c \le y \le d,\ \psi_1(y) \le x \le \psi_2(y)\}, \tag{5.16}$$

where ψ_1 and ψ_2 are continuous, piecewise smooth functions on $[c, d]$.