quite simple. We shall say that the region S is **x-simple** if it is the region between the graphs of two functions of x, that is, if it has the form

$$S = \{(x, y) : a \le x \le b,\ \varphi_1(x) \le y \le \varphi_2(x)\}, \tag{5.15}$$

where φ_1 and φ_2 are continuous, piecewise smooth functions on $[a, b]$. Likewise, we say that S is **y-simple** if it has the form

$$S = \{(x, y) : c \le y \le d,\ \psi_1(y) \le x \le \psi_2(y)\}, \tag{5.16}$$

where ψ_1 and ψ_2 are continuous, piecewise smooth functions on $[c, d]$.

Green's theorem: $\displaystyle\int_{\partial S} P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

oriented boundary

to define the orientation we need

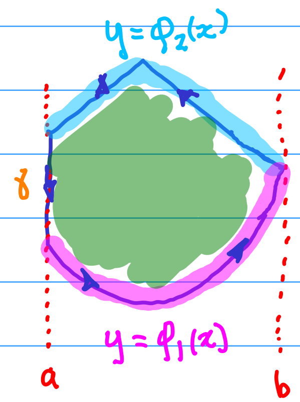
S Compact and $S = \overline{S^{int}}$

5.12 Theorem (Green's Theorem). *Suppose S is a regular region in $\mathbb{R}^2$ with piecewise smooth boundary ∂S. Suppose also that $\mathbf{F}$ is a vector field of class C^1 on $\overline{S}$. Then*

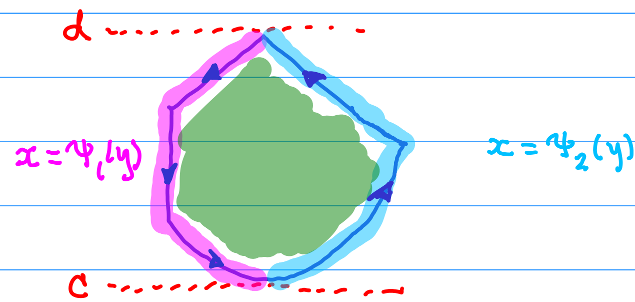
parameterized by piecewise smooth curve...

Consider first a simple region i.e. both x and y simple.

x-simple



y-simple



in positive orientation
φ_1 is a to b
φ_2 is b to a

in positive orientation
ψ_1 is d to c
ψ_2 is c to d

$$\int_{\partial S} P\,dx = \int_a^b P(x, \varphi_1(x))\,dx + \int_b^a P(x, \varphi_2(x))\,dx + \int_\gamma P\,dx$$

since γ is vertical then $dx = 0$

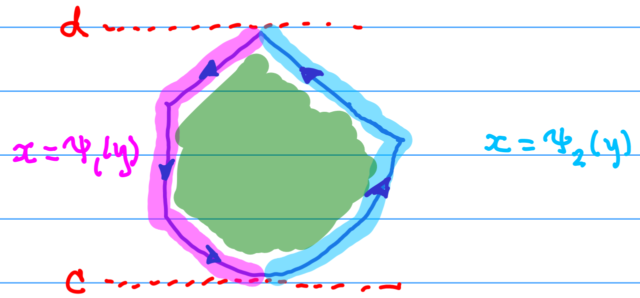
$$= \int_a^b \Big(P(x, \varphi_1(x)) - P(x, \underbrace{\varphi_2(x))}_{\text{y variable.}}\Big)\,dx$$

by fundamental theorem in One variable.

$$= \int_a^b \left(\int_{\varphi_2(x)}^{\varphi_1(x)} \frac{\partial P}{\partial y}(x,y)\, dy \right) dx$$

$$= - \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial P}{\partial y}(x,y)\, dy\, dx = - \iint_S \frac{\partial P}{\partial y}\, dA$$

y-simple



$$\int_{\partial S} Q\,dy = \int_c^d Q(\psi_2(y), y)\, dy + \int_d^c Q(\psi_1(y), y)\, dy$$

$$= \int_c^d \Big(\overbrace{Q(\psi_2(y), y) - Q(\psi_1(y), y)}^{x \text{ variable}} \Big)\, dy$$

$$= \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} \frac{\partial Q}{\partial x}(x,y)\, dx \right) dy = \iint_S \frac{\partial Q}{\partial x}\, dA$$

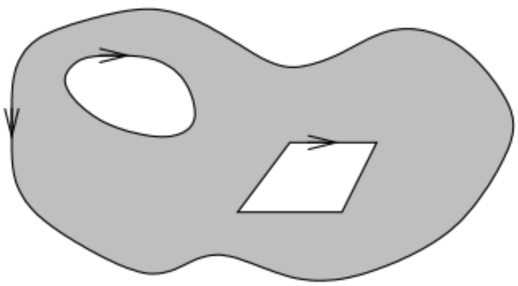
Therefore

$$\int_{\partial S} P\,dx + \int_{\partial S} Q\,dy = - \iint_S \frac{\partial P}{\partial y}\, dA + \iint_S \frac{\partial Q}{\partial x}\, dA$$

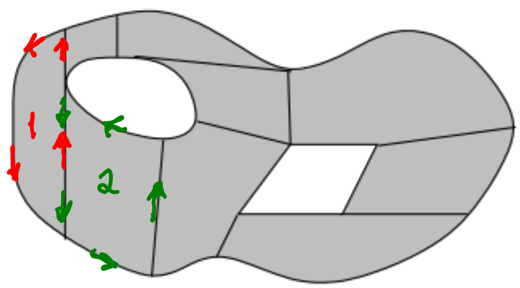
or

$$\int_{\partial S} P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Extension: Here is a region that's not simple



divide a non-simple regular region into simple regions



$J = 9$ simple regions in this case (finite number of pieces...)

Because the regions which share a boundary orient the boundary in opposite directions, then the shared boundaries don't contribute to the total line integral

$$\int_{\partial S} P\,dx + Q\,dy = \sum_{j=1}^{J} \int_{\partial S_j} P\,dx + Q\,dy$$

simple regions

$$= \sum_{j=1}^{J} \iint_{S_j} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{S} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

sums are finite otherwise trouble with limits and infinite sums.

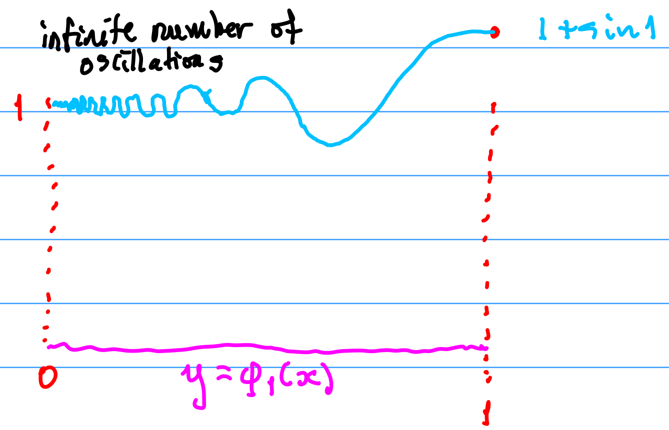
Example of a regular region

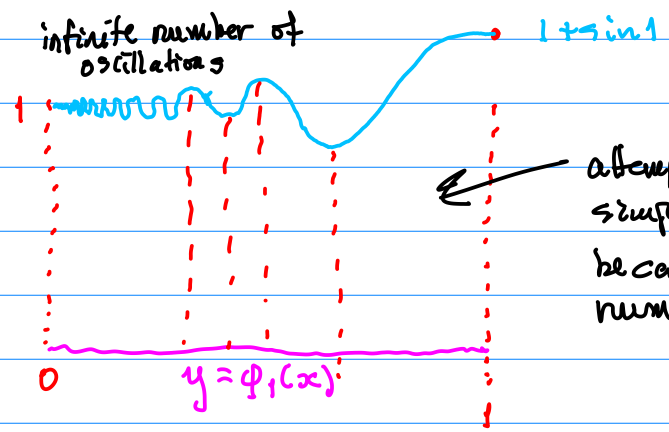
$$S = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1 + x^3 \sin x^{-1}\}$$

this is x-simple just by how it's defined

$\phi_1(x) = 0$ $\qquad$ $\phi_2(x) = 1 + x^3 \sin \frac{1}{x}$

more oscillations as $x \to 0$

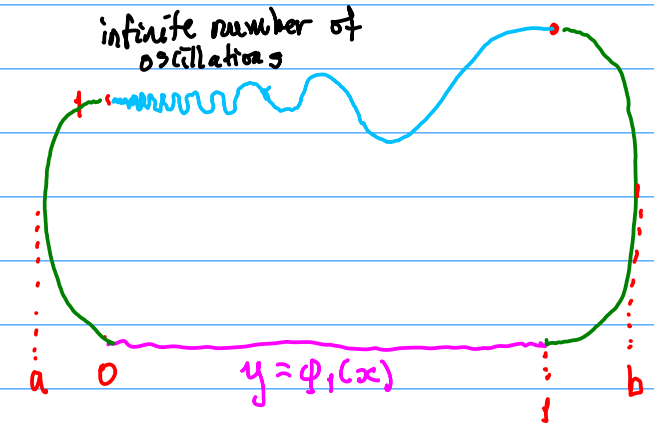




attempt to divide this into simple regions ... fails because I get an infinite number of pieces...

The way around this is in the Appendix Theorem B28 and the main hypothesis used is S being compact.

Assume the boundary is C^1 to start with...



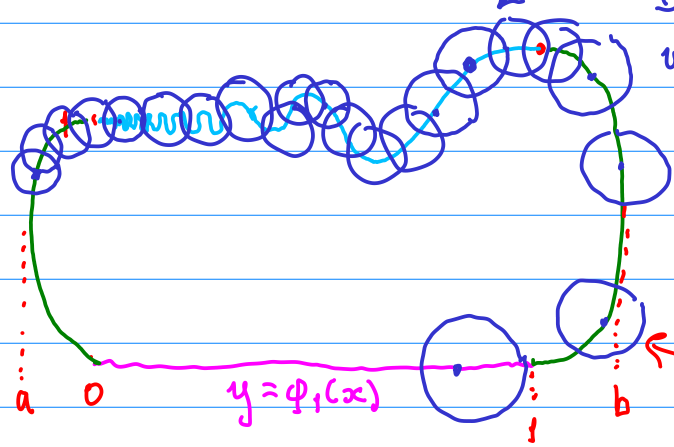
still can't divide into finite simple regions

at any point on the boundary I can find a neighborhood where the curve is given by either

$y=f(x)$

or

$x=g(y)$.



each neighborhood

$B_i = \{ x : |x - x_i| < \varepsilon_i \}$

where $x_i \in \partial S$ and $\varepsilon_i > 0$.

open set

finite because ∂S is compact

$$\partial S \subseteq \bigcup_{i=1}^{N} B_i$$

S^{int} is also an open set

$$S = S^{int} \cup \partial S = S^{int} \cup \bigcup_{i=1}^{N} B_i = \bigcup_{i=1}^{J} U_i$$

where $U_1 = S^{int}$ $\quad U_{i+1} = B_i$ $\quad J = N+1$.

This is an overlapping partition of the set. The B_i are simple and we'll be able to manage S^{int} with a partition of unity.

Partition of unity...

B.27 Theorem. *Suppose $K \subset \mathbb{R}^n$ is compact and $U_1, \ldots, U_J$ are open sets such that $K \subset \bigcup_1^J U_j$. Then there exists a finite collection $\{\varphi_m\}_1^M$ of C^∞ functions such that*

a. *the support of each φ_m is compact and contained in one of the sets U_j, and*
b. *$\sum_1^M \varphi_m(\mathbf{x}) = 1$ for all $\mathbf{x} \in K$.*

read this theorem for next time.